Rutgers University - Graduate Program in Mathematics

Written Qualifying Examination

Fall 1998

Day 1

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

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First Day – Part I: Answer each of the following three questions.

1. Suppose μ^* is an outer measure on X and $A_i, i = 1, 2, \ldots$, is a countable sequence of disjoint μ^*-measurable sets. Show that for any $E \subseteq X$,

$$\mu^*(E \cap (\cup_i A_i)) = \sum_i \mu^*(E \cap A_i).$$

2. Explain carefully how to evaluate

$$\int_0^\infty \frac{\cos x}{(A + x^2)^2}\, dx$$

for $A > 0$ using the Residue Theorem. What happens to the value of this integral as $A \to \infty$? What happens to the value of this integral as $A \to 0^+$?

3. (a) Suppose that G is a simple group and that G has a subgroup H such that $[G : H] = n > 1$. Show that G is isomorphic to a subgroup of the symmetric group S_n on n letters.
(b) Prove that there is no simple group of order 300.

First Day – Part II: Answer <u>three</u> out of the following six questions.

4. Let A, B be $n \times n$ real matrices. Suppose that A has n distinct real eigenvalues and that $AB = BA$. Show that B can be expressed as a polynomial in A.

5. Explicitly describe a function meromorphic in all of the complex plane $\mathbb{C}$ whose set of poles is exactly all the Gaussian integers: $\{m + ni \mid m \;\&\; n \text{ are integers}\}$. Prove that your function has the desired properties.

6. The Cantor set C is the subset of the unit interval consisting of those $x \in [0, 1]$ such that x has a base-3 expansion $\sum a_j 3^{-j}$ with $a_j \neq 1$ for all j. Alternatively $C = \bigcap_{j=1}^{\infty} C_j$, where $C_0 = [0, 1]$ and C_{j+1} is obtained from C_j by deleting the open middle-third of each subinterval of C_j. Prove that
(a) C is compact, totally disconnected, and nowhere dense.
(b) C has measure zero.
(c) C is uncountable.

7. (a) Show that the principal ideal (3) is not a prime ideal in $\mathbb{Z}[\sqrt{-5}]$.
(b) Find a prime ideal P such that $(3) \subset P \subset \mathbb{Z}[\sqrt{-5}]$.

8. Let m and μ be, respectively, the Lebesgue measure and the counting measure on the Borel σ-algebra on $[0, 1]$.
(a) Show that $m << \mu$ but $dm \neq f d\mu$ for any f.
(b) State the Lebesgue-Radon-Nikodym Theorem for signed measures, and explain why part (a) is not a counter-example!

9. (a) Define $T_t f(z) = f(z + t)$. Suppose that f is an entire function and that $T_t f \to f$ <u>uniformly</u> for all $z \in \mathbb{C}$ as $t \to 0$. Prove that f is a linear function.
(b) Give an example of a continuous function g defined on $\mathbb{R}$ which is <u>not</u> linear and which has the property that $T_t g \to g$ uniformly for all $x \in \mathbb{R}$ as $t \to 0$.

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Day 2

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

Second Day – Part I: Answer each of the following three questions.

1. Let μ be a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Assume that μ is translation invariant and that $\mu([0,1]) < \infty$. Prove that μ is a multiple of the Lebesgue measure. By giving a counterexample, show that the condition $\mu([0,1]) < \infty$ is essential.

2. Suppose that A and B are 6×6 nilpotent matrices over some field F. Assume that A and B have the same rank and the same minimal polynomial. Show that A and B are similar. Would the same be true for 7×7 nilpotent matrices?
[Hint: consider the Jordan normal forms of A and B.]

3. (a) Suppose that f is holomorphic in $\{z \in \mathbb{C} \mid |z| < 5\}$ and that $\sup\limits_{|z|<5} \{|f(z)|\} < 8$. Find $A > 0$ so that $\sup\limits_{|z|<3} \{|f'(z)|\} \leq A$ for all such f. Prove that your bound is valid.
(b) Find an example of a real-valued differentiable function g defined in the interval $(-5, 5)$ so that $\sup\limits_{-5<x<5} \{|g(x)|\} < 8$ and $\sup\limits_{-3<x<3} \{|g'(x)|\} \geq 1{,}000$.

Second Day – Part II: Answer <u>three</u> of the following six questions.

4. Let m be the Lebesgue measure on the real line $\mathbb{R}$. Suppose $f_i \to f$, $g_i \to g$ a.e. with f_i, g_i, f, $g \in L^1(\mathbb{R})$. Also suppose that $|f_i| \leq |g_i|$, and that $\int_{\mathbb{R}} g_i \, dm \to \int_{\mathbb{R}} g \, dm$. Prove that $\int_{\mathbb{R}} f_i \, dm \to \int_{\mathbb{R}} f \, dm$.

5. Prove that if F is a finite field, then every element of F can be expressed as a sum of two squares.

6. Let F_b be the entire function defined by $F_b(z) = z^4 + 100z^2 + e^{-b(z+2)}$. Prove that if b is a sufficiently large positive real number, then F_b has two roots (counting multiplicity) inside the unit circle.

7. Find and classify all finite singularities of the complex function

$$f(z) = \frac{\cos z - z - \frac{\pi}{2}}{\sin z + 1}.$$

8. Suppose that G_1, G_2 are finite groups such that $G_1 \neq 1$ and $G_2 \neq 1$. Prove that the following statements are equivalent.
(a) Whenever H is a subgroup of the direct product $G_1 \times G_2$, then there exist subgroups H_1, H_2 of G_1, G_2 respectively such that $H = H_1 \times H_2$.
(b) The orders $|G_1|$ and $|G_2|$ are relatively prime.

9. Let m be the Lebesgue measure on the real line $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be positive functions such that $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, where $1/p + 1/q = 1$ and $1 < p < \infty$, $1 < q < \infty$. Suppose that $\int_{\mathbb{R}} fg \, dm = ||f||_p ||g||_q$. Prove that there is a constant C such that $f(x)^p = Cg(x)^q$ for almost all x.