**Rutgers University - Graduate Program in Mathematics**

Written Qualifying Examination

Fall 1999

Day 1

This exam will be given in two three-hour sessions, one today and one tomorrow. At each session the exam will have two parts. Answer all three of the questions on Part I and three of the six questions on Part II. If you work on more than three questions on Part II, indicate clearly which three should be graded.

**First Day—Part I: Answer each of the following three questions.**

**1.** Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients, of strictly positive degrees $d_1$ and $d_2$, respectively. Let $r(x)$ be a polynomial of degree less than $d_1 + d_2$. Show that the rational function $\dfrac{r(x)}{p(x)q(x)}$ has a *partial fraction* decomposition of the form

$$\frac{a(x)}{p(x)} + \frac{b(x)}{q(x)} ,$$

where $a(x)$ and $b(x)$ are polynomials of degrees less than $d_1$ and $d_2$, respectively.

**2.** A function $f : \mathbb{C} \to \mathbb{C}$ is said to satisfy a *(global) Lipschitz condition* if there exists a constant $K$ for which $|f(z) - f(w)| \le K|z - w|$ for all $z, w \in \mathbb{C}$. Show that an entire (i.e., holomorphic) function that satisfies a global Lipschitz condition must have the form $f(z) = az + b$ for some complex constants $a$ and $b$.

**3.** Let $U \subset \mathbb{R}$ be a (possibly infinite) open interval and $f : U \to \mathbb{R}$ a differentiable function whose derivative is nondecreasing: $\xi_1 < \xi_2 \Rightarrow f'(\xi_1) \le f'(\xi_2)$. Prove carefully that for every $[a, b] \subset U$,

(a) $\max\{\, f(x) \mid a \le x \le b \,\} = \max\{\, f(a), f(b) \,\}$;

(b) If $\ell(x)$ is the linear function such that $\ell(a) = f(a)$ and $\ell(b) = f(b)$, then $f(x) \le \ell(x)$ holds for all $a \le x \le b$.

**First Day—Part II: Answer <u>three</u> of the following six questions. If you work on more than three questions, indicate clearly which three should be graded.**

**4.** Suppose that the finite group $G$ acts on the finite set $\Omega$. Let $n = |G|$ and $m = |\Omega|$. Assume that $G$ has exactly $r$ orbits on $\Omega$, and that their cardinalities are $m_1, \ldots, m_r$. An element $\alpha \in \Omega$ is chosen randomly, and independently of this an element $g \in G$ is chosen randomly, with all choices equally likely in each case. Determine, with a proof, the probability that $g\alpha = \alpha$.

**5.** Suppose that $g$ is a bounded measurable function defined on $\mathbb{R}$, and that for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, $f(z)$ is defined by

$$f(z) = \int_{-\infty}^{\infty} \frac{g(y)}{1 + zy^2}\, dy.$$

Prove that $f$ is holomorphic in the right half plane $\{\, z \in \mathbb{C} \mid \operatorname{Re} z > 0 \,\}$.

**6.** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \dfrac{x^2 y}{x^2 + y^2}$ for $(x,y) \neq (0,0)$, $f(0,0) = 0$.

(a) Prove that $f$ is continuous on $\mathbb{R}^2$ and that $(D_1 f)(x,y) \equiv \frac{\partial f}{\partial x}(x,y)$ and $(D_2 f)(x,y) \equiv \frac{\partial f}{\partial y}(x,y)$ exist for all $(x,y) \in \mathbb{R}^2$ and are bounded on $\mathbb{R}^2$.

(b) Prove that for every unit vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, the directional derivative $(D_{\mathbf{u}} f)(0,0) \equiv \lim\limits_{t \to 0} \dfrac{f(t\mathbf{u}) - f(0,0)}{t}$ exists.

(c) Prove that $f$ is *not* differentiable at $(0,0)$.

**7.** Define the *exponential* $e^A$ of a square matrix $A$ with complex entries by the standard power series:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \; .$$

Show:

(a) For any such matrix $A$, this infinite series converges.

(b) If $A$ and $B$ are similar matrices, then $e^A$ and $e^B$ are similar matrices.

(c) $\det(e^A) = e^{\operatorname{tr}(A)}$. (Here *tr* denotes the trace.)

**8.** Suppose that $f(z)$ is holomorphic in the unit disc $D = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$, that $f(0) = 0$, and that $\operatorname{Re}[f(z)] < 1$ for $z \in D$. Show that

$$|f(z)| \leq \frac{2|z|}{1 - |z|}$$

for all $z \in D$.

**9.** Evaluate the following limits rigorously, justifying your answers with the aid of appropriate theorems.

(a)

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, dx.$$

(Hint: Prove that $\left(1+\frac{x}{n}\right)^{-n} \le \left(1+x+\frac{x^2}{4}\right)^{-1}$ for $x \ge 0$ and $n \ge 2$.)

(b)

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^{n} e^{x/2}\, dx.$$

(Hint: Prove that $\left(1-\frac{x}{n}\right)^{n} \le e^{-x}$ for $n \ge 1$ and $0 \le x \le n$.)

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Day 2

This is the second of two three-hour exam sessions; you should have taken the first session yesterday. The exam has two parts. Answer all three of the questions on Part I and three of the six questions on Part II. If you work on more than three questions on Part II, indicate clearly which three should be graded.

**Second Day—Part I: Answer each of the following three questions.**

**1.** This problem concerns Lebesgue integration of functions from $\mathbb{R}$ to $\mathbb{R}$. For such functions, a trivial form of Hölder's inequality says that if $f \in L^\infty(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ then

$$\left| \int_{\mathbb{R}} fg \right| \leq \|f\|_\infty \|g\|_1. \tag{$*$}$$

(a) Prove that for each $g \in L^1(\mathbb{R})$ there is an $f \in L^\infty(\mathbb{R})$ with $\|f\|_\infty = 1$ such that equality holds in $(*)$.

(b) Prove that for each $f \in L^\infty(\mathbb{R})$ and each $\epsilon > 0$ there is a $g \in L^1(\mathbb{R})$ with $\|g\|_1 = 1$ such that

$$\left| \int_{\mathbb{R}} fg \right| \geq \|f\|_\infty - \epsilon.$$

**2.** Let $P(z)$ be a polynomial in $z$ of degree two or higher. Let $\mathcal{C}_R$ be the positively oriented semicircle of radius $R$, with center at the origin, in the upper half plane. Show that

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{e^{ikz}}{P(z)}\, dz = \begin{cases} 0, & \text{if } k \geq 0, \\ 2\pi i \displaystyle\sum_{n=1}^{m} \operatorname{Res}_{\zeta_n} \frac{e^{ikz}}{P(z)}, & \text{if } k < 0, \end{cases}$$

where $\zeta_1, \ldots, \zeta_m$ are the distinct zeros of $P(z)$ and $\operatorname{Res}_\zeta f(z)$ is the residue of $f(z)$ at $\zeta$.

**3.** Let $G$ be a group and let $H$ be the subgroup of $G$ generated by the set $\{\, x^2 \mid x \in G \,\}$ of all squares in $G$.

(a) Show that $H$ is a normal subgroup of $G$.

(b) Show that $G/H$ is abelian.

(c) Let $a$, $b$, $c$, and $d$ be elements of $G$. Using (b), deduce that

$$abcdcccbadacdcad$$

is an element of $H$.

**Second Day—Part II: Answer three of the following six questions. If you work on more than three questions, indicate clearly which three should be graded.**

**4.** A function $f : \mathbb{R} \to \mathbb{R}$ is called 1-periodic if $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Let $V$ be the vector space of all 1-periodic $C^\infty$ functions $f : \mathbb{R} \to \mathbb{R}$, let $h : \mathbb{R} \to \mathbb{R}$ be a fixed 1-periodic $C^\infty$ function, and consider the linear differential operator $T : V \to V$ defined by

$$(Tf)(x) = f''(x) + h(x)f(x).$$

Show that any finite-dimensional $T$-invariant subspace of $V$ is the span of certain one-dimensional $T$-invariant subspaces. (Hint: use the inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$.) Recall that a subspace $W$ of $V$ is $T$-invariant if and only if $T(W) \subseteq W$.

**5.** Suppose that $a_k = \displaystyle\int_0^1 \frac{\cos k\pi x}{\sqrt{x}}\,dx$ for $k = 0, 1, 2 \ldots$. Show that $\displaystyle\sum_{k=0}^{\infty} a_k^2 = \infty$.

**6.** Let $P(z)$ be a polynomial, with $P(0) \neq 0$, whose complex zeros, enumerated according to multiplicity, are $\{a_k\}_{k=1}^n$. Find expressions for the sums $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k}$ and $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k^2}$ in terms of $P(0)$, $P'(0)$, and $P''(0)$. (Hint: Begin by finding an expression for $\displaystyle\sum_{k=1}^{n} \frac{1}{z - a_k}$.)

**7.** A metric space is called *separable* if it contains a countable dense set. Let $A$ be a subset of a separable metric space $(X, d)$. Prove that $A$ (with the metric inherited from $X$) is separable.

**8.** This problem concerns polynomials and rational functions in two variables, $t$ and $u$, over the complex field $\mathbb{C}$. Suppose that

$$f(t,u) = \sum_{i=0}^{n} \alpha_i(u) t^i \qquad \text{and} \qquad g(t,u) = \sum_{j=0}^{m} \beta_j(u) t^j\,,$$

where all $\alpha_i$ and $\beta_j$ are rational functions of $u$ ($\alpha_i, \beta_j \in \mathbb{C}(u)$) and $\alpha_n(u) = \beta_m(u) = 1$. Show that

$$fg \in \mathbb{C}[t,u] \iff f \in \mathbb{C}[t,u] \text{ and } g \in \mathbb{C}[t,u].$$

**9.** Let $\theta : [0,1] \to [0,1]$ be a strictly increasing, continuously differentiable function with $\theta(0) = 0$ and $\theta(1) = 1$. In this problem we write $V[F; a, b]$ for the total variation of the function $F$ over the interval $[a, b]$.

(a) Let $f : [0,1] \to [0,1]$ be a function of bounded variation and let $g = \theta \circ f \circ \theta^{-1}$. Show, directly from the definition of total variation, that there exists a constant $K$ depending only on $\theta$ such that

$$V[g; 0, 1] \leq K\, V[f; 0, 1].$$

(b) Determine, with a proof, the constant $K$ in (a) which is optimal in the sense that for any any $K' < K$ there exists an $f$ with $V[g; 0, 1] > K'\, V[f; 0, 1]$.