RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2002, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Suppose that G is a finite group of even order and e is the identity element of G. Prove that G contains an element $a \neq e$ such that $a^2 = e$.

2. Find $\displaystyle\int_0^\infty \frac{dx}{x^{1/4}(1+x)}$.

3. Let X be a closed subset in $\mathbb{R}^n$ with the standard metric and suppose that $\displaystyle\bigcup_{k\in\mathbb{Z}} kX = \mathbb{R}^n$. Prove that X contains some nonempty open ball.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let V be a complex vector space of finite dimension n. If $T\colon V \to V$ is a linear transformation whose minimal polynomial has degree m, show that there exists a linearly independent set $\{v_1, \ldots, v_d\}$ of eigenvectors of T such that $n \leq md$.

5. Call an integral domain R **special** if and only if the intersection of any two principal ideals in R is again principal, that is, generated by one element.

a) Show that if R is special, then any two nonzero elements of R have a greatest common divisor (g.c.d.).

b) Give an example of a special integral domain R and nonzero elements $a, b \in R$ such that the g.c.d. of a and b is not an R-linear combination of a and b.

6. Suppose the complex function $f(z)$ is analytic inside the circle $|z| < 2$ and that $f(z)$ is real for all z such that $|z| = 1$. Prove that f is a constant function.

7. Suppose that S_2 is the open square region defined by

$$S_2 = \{z \in \mathbb{C} : |\mathrm{Re}\ z| < 2 \text{ and } |\mathrm{Im}\ z| < 2\}$$

and S_1 is the open square region defined by

$$S_1 = \{z \in \mathbb{C} : |\mathrm{Re}\ z| < 1 \text{ and } |\mathrm{Im}\ z| < 1\}.$$

Let $S = S_2 \setminus \overline{S_1}$, a square with a square hole. Show that there is no one-to-one conformal mapping of the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto S.

8. Prove or disprove: if $a, b \in \mathbb{R}$, $a < b$, and $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous functions, then the product $f \cdot g$ is absolutely continuous.

(A function h is *absolutely continuous* on $[a, b]$ if, given $\varepsilon > 0$, there is $\delta > 0$ so that if $a \leq x_0 < x_1 < \ldots < x_n \leq b$ are such that $\sum_{i=0}^{n-1} |x_{i+1} - x_i| < \delta$ then $\sum_{i=0}^{n-1} |h(x_{i+1}) - h(x_i)| < \varepsilon$.)

9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \int_{-\infty}^{+\infty} \frac{\sin(t + e^x)}{1 + t^2}\, dt.$$

Prove that f is continuous.

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Spring 2002, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. a) Find all invertible elements in the ring $\mathbb{Z}[i]$ of integral complex numbers.

b) Which of the elements $2, 3, 5$ in $\mathbb{Z}[i]$ are prime elements? Explain your answer.

2. Prove or disprove the following statement:

   There exists a real-valued function $f(x)$ defined for all $x \in \mathbb{R}$ such that $f(x)$ is continuous and differentiable at $x = 0$ and discontinuous at all $x \neq 0$.

3. Let $\lambda > 1$. Show that the equation

$$z = \lambda - e^{-z}$$

has exactly one root in the half-plane $\operatorname{Re} z > 0$ and that this root is a real root.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose that G is any group of order 36, and G acts transitively on sets Ω and Ψ such that $|\Omega| = 12$ and $|\Psi| = 4$. Show that there is a function $f\colon \Omega \to \Psi$ such that $f(g \cdot \omega) = g \cdot (f(\omega))$ for all $\omega \in \Omega$ and all $g \in G$.

5. Let R be a ring. Prove or disprove that a finitely generated R-module is necessarily finitely generated as an abelian group.

6. Prove that the closed unit interval is not homeomorphic to the closed unit square.

7. Suppose that f and g are entire functions with $|f(z)| \leq |g(z)|$. Prove that $f(z) = cg(z)$ for some constant c.

8. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued continuously differentiable functions on the unit interval, $[0, 1]$, with $f_n(0) = 0$ for all $n \in \mathbb{N}$. Suppose also that

$$\sup_{n\in\mathbb{N}} \int_0^1 \left(f_n'(x)\right)^2 \, dx < \infty \, .$$

Prove that there is a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ which converges uniformly on $[0, 1]$.

9. Prove or disprove: if K and M are metric spaces, $f\colon K \times M \to R$ is a continuous function, and K is compact, then the function $g\colon M \to R$ defined by

$$g(x) = \max\{f(k, x) : k \in K\}$$

is continuous.