RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2003, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function such that $\int_0^1 x^k f(x)\,dx = 0$ for all nonnegative integers k (that is, for $k = 0, 1, 2, \ldots$). Prove that $f(x) = 0$ almost everywhere.

2. Let K be a compact metric space. Prove that there exists a dense finite or countably infinite subset $A \subseteq K$.

3. Let F be a field and let $GL_6(F)$ be the set of all invertible 6×6 matrices over F, which can be regarded as the set of all invertible linear maps from F^6 to F^6. Let C be the set of conjugacy classes of $GL_6(F)$, and let $m(x)$ be the polynomial

$$m(x) = (x - 1)^2(x + 1)^2$$

Let C_m be the set of classes $S \in C$ such that the minimal polynomial of the members of S is m. Find representatives of each of the clases belonging to C_m. *Note*: F could be of characteristic two, in which case of course $m(x) = (x - 1)^4$.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is *convex* if $f(ax + (1-a)y) \le af(x) + (1-a)f(y)$ whenever $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $0 \le a \le 1$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is convex and $g : [0,1]^n \to \mathbb{R}$ is Lebesgue integrable, then

$$f\Big(\int_{[0,1]^n} g(x)\,dx\Big) \le \int_{[0,1]^n} f(g(x))dx\,.$$

You may use the fact that a convex function is continuous.

5. Assume that $z \to f(z)$ is analytic on the open unit disk, i.e., for $|z| < 1$, continuous for $|z| \le 1$, and such that $|f(z)| = 1$ if $|z| = 1$. Show that f is a rational function.

6. Suppose that A is an $n \times n$ real matrix. Show that there is an $n \times n$ real matrix B such that $A = B^{\mathrm{T}}B$ if and only if A is symmetric and all eigenvalues of A are non-negative. Here B^{T} denotes the transpose of B.

7. Let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ be an infinite sequence of connected and closed subsets of the plane, $\mathbb{R}^2$. Is it true that $\cap_{i=1}^{\infty} X_i$ is connected? What if we assume that X_1 is compact?

8. Show that there does not exist any function f which is analytic in $\mathbb{C}$ minus a (denumerable) set of *isolated singularities* and such that $f(e^z) = z$ wherever $f(e^z)$ is defined.

9. Suppose that G is a finite group with the property that every maximal subgroup of G is normal in G. Let p be a prime and P be a Sylow p-subgroup of G. Show that P is a normal subgroup of G.

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2003, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let $a \in \mathbb{R}^+$. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(at)}{(t^2+1)^2} dt$$

and briefly justify the steps involved in the calculation.

2. Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the induced topology, is a connected space.

3. Let R be a principal ideal domain and F its field of fractions. Let M be a finitely generated R-submodule of the R-module F/R. Show that M is a cyclic R-module, i.e., that there is an element r of R such that M is isomorphic to $R/(r)$, where (r) is the principal ideal generated by r.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Prove that

$$\lim_{\alpha\to+\infty}\int_0^{+\infty}\frac{\sin(\alpha x)}{1+x^2}\,dx \;=\; 0\,.$$

5. Suppose that R is a finite-dimensional algebra over a field F. Assume that R satisfies the cancellation law, i.e., if $x,\, y \in R$ and $xy = 0$, then either $x = 0$ or $y = 0$. Show that R is a division ring. *Note:* It is not assumed that R has an identity or that R is commutative.

6. Assume f is a meromorphic function on $\mathbb{C}$ and has at least one pole. Show that the function $f \circ f$ (i.e., the composition of f with itself) is meromorphic on $\mathbb{C}$ *iff* f is rational.

7. Let G be a finite subgroup of $GL_2(\mathbb{R})$, the group of invertible 2×2 real matrices.

(a) Show that every matrix in G must be diagonalizable over the *complex* numbers.

(b) Show that unless $G = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$ there exists a matrix in G with non-positive trace.

8 Let u be a continuous real-valued function on the closed unit disk $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$ which is of class C^∞ on the open disc $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$, satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

in D, and is such that $u(x,y) = x^2 - y^2$ whenever $x^2+y^2 = 1$. Prove $u(x,y) = x^2-y^2$ whenever $x^2+y^2 \le 1$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Let

$$I(f) = \left\{p \in [1,\infty) : \int_{-\infty}^{+\infty} |f(x)|^p\,dx < \infty\right\}.$$

Prove that $I(f)$ is an interval. (Note: the empty set *is* an interval.)