RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2004, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. For an integer n, let $H(n)$ denote the additive subgroup of $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ generated by $(4, n)$ and $(8, 12)$ and let $G(n)$ denote the quotient group of $\mathbb{Z}^2$ by $H(n)$.

(a) For what value(s) of n is $G(n)$ infinite?

(b) For what value(s) of n is $G(n)$ a cyclic group?

(c) What is the smallest finite order that $G(n)$ can have?

2. Use the residue theorem to evaluate

$$\int_0^\infty \frac{1}{(x^2 + a^2)^2}\, dx, \qquad a > 0.$$

3. Let $\{f_n\}$ be a sequence of functions in $C^1([0,1])$ such that $f_n(0) = 0$ and $|f_n'(x)| \le g(x)$, where $g(x)$ is a Lebesgue integrable function on $[0,1]$. Prove that there is a subsequence of $\{f_n\}$ that converges uniformly on $[0,1]$.

Exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let G be a group of order 56. Show that in G either a Sylow 2-subgroup is normal or a Sylow 7-subgroup is normal.

5. Let $f(z)$ be analytic on $|z| \leq 1$ and satisfy $|f(z)| < 1$ if $|z| = 1$. Show that f has a unique fixed point on $|z| < 1$, namely a unique point z_0 in the disk $|z| < 1$ such that $f(z_0) = z_0$.

6. Given functions $\{f_n : [0,1] \to [-1,1]\}$ such that $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$ for all $0 \leq a < b \leq 1$, show that
a) For every Lebesgue measurable subset $A \subseteq [0,1]$, $\lim_{n\to\infty} \int_A f_n(x)dx = 0$,
b) For every Lebesgue measurable f on $[0,1]$ such that $\int_0^1 |f(x)|dx < \infty$, $\lim_{n\to\infty} \int_0^1 f(x)f_n(x)dx = 0$

7. Show that there is no continuous injective map $f : S^1 \to \mathbb{R}^1$.

8. Let $A, B \subset \mathbb{R}$ and suppose the outer Lebesgue measure of A is zero. Prove that the outer Lebesgue measure of $A \times B \subset \mathbb{R}^2$ is zero also.

9. Assume that $f(x)$ is defined and differentiable on $[0,\infty)$, that $f(0) = 0$, and that $f'(x)$ is decreasing. Show that $f(x)/x$ is decreasing.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2004, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let $f(X) = (X-1)(X+1)^2$ and $g(X) = (X-1)(X+1)(X+2)$ in $\mathbb{Q}[X]$. Find a 3-by-3 rational matrix A such that $g(A) = 0$ and the characteristic polynomial of A is $-f$. Here the characteristic polynomial of A is defined as $\det(A - XI)$.

2. Let f be a Lebesgue integrable function on $\mathbb{R}^2$, that is

$$\int_{\mathbb{R}^2} |f(x)|\, dx < \infty.$$

Assume that

$$\int_B f(x)\, dx = 0$$

for all square subsets $B \subset \mathbb{R}^2$. Prove that $f(x) = 0$ almost everywhere.

3. If X and Y are metric spaces with X locally compact and $f : X \to Y$ is a continuous surjection, must Y be locally compact? Prove or give a counterexample.

Exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. In the ring $\mathbb{Q}[X]$ of polynomials in one variable with rational coefficients, all ideals are principal. It follows from the Hilbert Basis Theorem that all ideals of $\mathbb{Z}[X]$ are finitely generated.

(a) Exhibit an ideal of $\mathbb{Z}[X]$ that is not principal. Justify your answer.

(b) Is there an upper bound on the minimum number of generators of an ideal in $\mathbb{Z}[X]$? Justify your answer.

5. Determine the order of the largest finite group with exactly two conjugacy classes of elements.

6. Find an open, connected set G in the complex plane and two distinct, continuous, complex-valued functions f and g defined on G such that $f(z)^2 = g(z)^2 = 1 - z^2$ for all $z \in G$. Are f and g analytic? Can you make G maximal?

7. Let f be a one-to-one analytic mapping of the disk $|z| < 1$ onto itself such that $f(1/2) = 0$ and $f(0) = 1/2$. Find the image under f of the part of the x-axis with $1/2 < x < 1$.

8. Let $A \subset \mathbb{R}$ be a Lebesgue-measurable set with finite measure $|A|$ and let f be a bounded, continuous function on $\mathbb{R}$. Show that

$$\left(\frac{1}{|A|}\int_A |f(x)|^p \, dx\right)^{1/p}$$

is an increasing function of p. What is the limit of this function as $p \to \infty$? Prove your answer.

9. If X is a topological space, let $DX = \{(x, x) \in X \times X : x \in X\}$. Show that X is Hausdorff if and only if DX is closed in $X \times X$.

Exam Day 2 End