RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2006, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).

If you work on four or more questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then only the first three questions attempted will be graded, as determined by the order in which they appear in the examination book(s).

First Day—Part I: Answer each of the following three questions

1. How many different Sylow 2-subgroups and Sylow 5-subgroups are there in a non-commutative group of order 20? Justify your answer.

2. Let $f(z) = z^{10} - 10z^3 + 2\sin z + 1$. Determine how many zeros f can have in the unit disk, and then prove your statement.

3. Suppose that E is a measurable set in $\mathbb{R}^n$ with measure $|E| < \infty$ and that f is a Lebesgue integrable function on E. Let $\{E_k\}$ be a sequence of measurable subsets of E such that

$$|\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k| = 0.$$

Prove that $\int_{E_k} f \to 0$ as $k \to \infty$.

First Day—Part II: Answer three of the following six questions. If you work on four or more questions, indicate clearly which three should be graded.

4. Let A be a complex matrix of order n. Prove that A is nilpotent if and only if all its eigenvalues are equal to zero.

5. Prove that, for any group G, the set of all automorphisms of the form $\phi(g) = xgx^{-1}$, $x \in G$ is a normal subgroup of the group of all automorphisms of G.

6. Let H be the "Hawaiian earring,"

$$H = \bigcup_{n=1}^{\infty} C_n$$

where C_n is the circle in the plane with center $(0, 1/n)$ and radius $1/n$. Show that H is a compact subset of the plane.

7. For $r > 0$, set $D_r := \{z \in \mathbb{C} : \ |z| < r\}$. Assume that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is holomorphic over the unit disk D_1 but that f does not extend holomorphically to D_r for any $r > 1$. Suppose that a_j is a positive real number for each $j \geq 0$. Prove that $f(z)$ does not extend holomorphically across $z = 1$.

8. Let f be a Lebesgue integrable function on $\mathbb{R}$, $f \in L^1(\mathbb{R})$. Show that if $\int_I f = 0$ for all intervals $I \subseteq \mathbb{R}$ then $f = 0$ almost everywhere.

9. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for any $x, y \in \mathbb{R}$, and all $0 \leq t \leq 1$. Prove that if f is continuous and convex, and if $g \in L^1(E)$ for some $E \subset \mathbb{R}$ with $|E| < \infty$, then:

$$f\left(\frac{1}{|E|}\int_E g(x)\,dx\right) \leq \frac{1}{|E|}\int_E f\big(g(x)\big)\,dx.$$

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2006, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).

If you work on four or more questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then only the first three questions attempted will be graded, as determined by the order in which they appear in the examination book(s).

Second Day—Part I: Answer each of the following three questions

1. Let $\mathbb{Q}[x]$ denote the vector space of polynomials over $\mathbb{Q}$ and $\mathbb{Q}[[x]]$ denote the vector space of power series $\sum_{i=0}^{\infty} a_i x^i$ over $\mathbb{Q}$. Give an explicit isomorphism between the vector space $\mathbb{Q}[[x]]$ and the dual of the vector space over $\mathbb{Q}[x]$ and prove that it is an isomorphism.

2. Let D be the intersection of the following two domains $D_1 := \{z : |z+1| < 2\}$ and $D_2 := \{z : |z-1| < 2\}$. Write down a conformal map from D to the unit disk.

3. A Hausdorff topological space X is called *locally compact* if each point $x \in X$ lies in an open subset whose closure is compact. Prove or disprove the assertion that every subset X of Euclidean space $\mathbb{R}^n$ is locally compact.

Second Day—Part II: Answer three of the following six questions. If you work on four or more questions, indicate clearly which three should be graded.

4. Prove that every group of order 6 is either isomorphic to the cyclic group $\mathbb{Z}/6\mathbb{Z}$ or to the symmetric group S_3.

5. Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0,1]$, and that $f_n \geq 0$ almost everywhere for each n. Assume that for all n,

$$\int_0^1 f_n(x)dx = 1, \quad \text{and} \quad \int_{1/n}^1 f_n(x)dx < 1/n.$$

Let $g(x) = \sup_n\{f_n(x)\}$. Show that

$$\int_0^1 g(x)dx = +\infty.$$

6. Define a metric $d(x,y)$ on $\mathbb{R}$ by setting $d(x,y) = |e^x - e^y|$. Determine if $\mathbb{R}$ with this metric is complete. Justify your answer.

7. Prove that the ring $\mathbb{C}[[x]]$ of complex power series is a principal ideal domain.

8. Calculate the following integral (showing your work): $\displaystyle\int_0^\infty \frac{x^{1/4}}{1+x^3}\,dx.$

9. Let $g(x)$ be a nonnegative Lebesgue measurable function on a measurable subset $E \subseteq \mathbb{R}^n$ and let ω_g be the function $\omega_g(x) = |\{t \in E : g(t) > x\}|$. Suppose that $f(x)$ is a continuously differentiable function on $[0,+\infty)$ satisfying $f(0) = 0$ and $\lim_{x\to+\infty} f(x)\,\omega_g(x) = 0$. Show that

$$\int_E f\big(g(t)\big)\,dt = \int_0^{+\infty} f'(x)\,\omega_g(x)\,dx.$$