RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2009, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Use the Residue Theorem to compute the improper integral

$$\int_0^\infty \frac{x \sin x}{1+x^2} dx.$$

Describe clearly the contour of integration and the residue computation. If you claim that the limit of some integral occurring in your calculation is zero, explain carefully why this is so.

2. Let m^* denote Lebesgue outer measure on subsets of $\mathbf{R}$. Show that if A and B are *any* two subset of R a positive distance apart, then $m^*(A \cup B) = m^*(A) + m^*(B)$. Two subsets of $\mathbf{R}$ are a positive distance apart if $d(A, B) = \inf\{|x - y|;\, x \in A,\, y \in B\} > 0$.

3. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing 56. (You may quote the Sylow theorems without proof.)

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose that $f(z)$ is a holomorphic function on all of $\mathbf{C}$. Assume that there are constants $A > 0$ and a non-negative integer k so that

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq Ar^k$$

for all $r > 0$. Prove that $f(z) = Cz^k$ for some constant C.

5. Find a bijective conformal map from $D := \{z : |z| < 1, Re(z) > 0\}$ onto the unit disk $\Delta := \{z \in \mathbf{C} : |z| < 1\}$. Exhibit your map as a composition of suitable rational and elementary transcendental functions. Show the transformations of domains made by each of your functions.

6. Compute the limit and justify the calculation. You may express the limit as an integral.

$$\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{(1+\frac{x}{n})^n}\, dx.$$

7. Assume that $f : (0, \infty) \to \mathbf{R}$ is Lebesgue measurable and that $\int_0^\infty \frac{|f(x)|}{x} dx < \infty$. Let $F(x) = \int_1^x f(t)dt$ and let $b > a > 0$. With a careful justification of your steps, show that

$$\frac{F(bx) - F(ax)}{x^2}$$

is integrable on $(0, \infty)$ and calculate $\int_0^\infty \frac{F(bx)-F(ax)}{x^2} dx$ in terms of a, b and $\int_0^\infty \frac{f(x)}{x} dx$.

8. Let V be vector space over $\mathbf{R}$ with a positive definite inner product $(,)$ and W is a finite dimensional subspace of V. If $x \notin W$, prove that there exists $y \in W^\perp$ such that $(x, y) \neq 0$.

9. Prove that the polynomial $x^6 + x^3 + 1$ cannot be written as a product of two polynomials of integer coefficients and positive degrees.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2009, Day 2

This examination will be given in two three-hour sessions, today's being the second part.
At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Prove that an $n \times n$ matrix A with complex number entries so that $A^n = A$ is diagonalizable.

2. Let $\{f_n\}$ be a sequence of non-negative, measurable functions on the measure space $(X, \mathcal{F}, \mu)$. Suppose that $f_n \to f$ in measure as $n \to \infty$. Show that $\int f \, d\mu \leq \liminf \int f_n \, d\mu$.

3. Suppose (X, d) is a compact metric space and $f : X \to \mathbf{R}$ is continuous. Show that for any $\epsilon > 0$, there exists a constant M so that

$$|f(x) - f(y)| \leq Md(x, y) + \epsilon$$

for all $x, y \in X$.

The exam continues on the next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $f(z) = \sum_{j=1}^{\infty} z^{n!}$. Show that $f(z)$ defines a holomorphic function over the unit disk $\Delta := \{z \in \mathbf{C} : |z| < 1\}$. Show that f does not extend continuously across any point on the unit circle $S := \{z \in \mathbf{C} : |z| = 1\}$.

5. If ν and ρ are measures on a measure space $(X, \mathcal{M})$, we say that $\nu \leq \rho$ if $\nu(A) \leq \rho(A)$ for all $A \in \mathcal{M}$. Assume that $\{\nu_n\}$ is a sequence of measures on $(X, \mathcal{M})$, such that $\nu_1 \leq \nu_2 \leq \cdots$. Define $\nu(A) := \lim_{n\to\infty} \nu_n(A)$, $A \in \mathcal{M}$.

(a) Show that ν defines a measure.

(b) Assume there exists a measure μ so that for each n, $\nu_n \ll \mu$. Show that $\nu \ll \mu$ also and $d\nu/d\mu = \lim_{n\to\infty} d\nu_n/d\mu$, μ-a.e. Here $\nu \ll \mu$ means that if $\mu(A) = 0$ then $\nu(A) = 0$ for any measurable set A.

6. Let $f : R \to R$ be a continuously differentiable function such that $\lim_{x\to\infty} \frac{f(x)}{x} = 1$. Assume $\lim_{x\to\infty} f'(x)$ exists. Prove directly that this limit must equal 1. (Invoking L'Hôpital's rule is not a permissible answer!)

7. Let R be a commutative ring with 2^n elements such that every element r of R satisfies $r^2 = r$. Prove that R is isomorphic to $\prod_{i=1}^{n} (\mathbf{Z}/(2\mathbf{Z}))$.

8. A module M over $\mathbf{Z}$ is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Determine all the irreducible modules over $\mathbf{Z}$.

9. Let $f : \mathbf{R} \to \mathbf{R}$ be a twice differentiable function so that $f(0) = 0$, $f'(0) > 0$ and $f''(x) \geq f(x)$ for all $x \geq 0$. Prove that $f(x) > 0$ for all $x > 0$.

Exam Day 2 End