RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2015

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Prove that the group $\mathbb{Q}$ of rationals under addition is a torsion free abelian group, but is not a free abelian group.

2. Let $\mathbb{Z}[x]$ denote the polynomial ring in the variable x with coefficients in $\mathbb{Z}$.

(a) Let $I \subset \mathbb{Z}[x]$ be the the ideal consisting of all elements whose constant term is 0. Prove that I is a prime ideal of $\mathbb{Z}[x]$ but is not a maximal ideal.

(b) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

3. Prove that a finite group G is the internal direct product of its Sylow subgroups if and only if every Sylow subgroup is normal in G.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Recall that the group $GL_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by the usual matrix-vector multiplication $A \cdot v = Av$, where $A \in GL_2(\mathbb{R})$ and v is a column vector in $\mathbb{R}^2$.

(a) Determine the number of orbits for this action, and describe each orbit.

(b) Find the pointwise stabilizer of the set $\{(x, y) \in \mathbb{R}^2 \mid y = x\, ,\ x \neq 0\}$.

5. Let $\rho : G \to GL_3(\mathbb{C})$ be a homomorphism, where G is the cyclic group of order 3. Show that with respect to some basis of $\mathbb{C}^3$, every element of $\rho(G)$ is a diagonal matrix having cube roots of unity on its diagonal.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. For $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, let

$$\begin{aligned} f(\mathbf{v}) &= x^2 + y^4 + z^6 - 3\,, \text{ and} \\ g(\mathbf{v}) &= x + y + z\,. \end{aligned}$$

Let $S = \{\mathbf{v} \in \mathbb{R}^3 : f(\mathbf{v}) = g(\mathbf{v}) = 0\}$. Prove that for every $\mathbf{v} \in S$, there exists

- A) an open subset $\Omega \subset \mathbb{R}^3$ containing $\mathbf{v}$,
- B) an open interval $(-\epsilon, \epsilon) \subset \mathbb{R}$, and
- C) a continuously differentiable, one-to-one map $\gamma : (-\varepsilon, \varepsilon) \to U$,

such that

- D) $\gamma'(t)$ does not vanish on $(-\varepsilon, \varepsilon)$, and
- E) γ's image is equal to $\Omega \cap S$.

2. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}\,, \qquad \text{for } a > 1\,.$$

3. Find a conformal (i.e., biholomorphic) map from the quarter-circle

$$Q = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0, \text{ and } |z| < 1\}$$

onto the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

Exam continues...

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Let f be a holomorphic function on $\mathbb{D}$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Prove that if f has at least two fixed points (that is, if there exist $a, b \in \mathbb{D}$ such that $a \neq b$, $f(a) = a$ and $f(b) = b$), then $f(z) \equiv z$ for all $z \in \mathbb{D}$.

5. Prove that if $u : \mathbb{C} \mapsto \mathbb{R}$ is a harmonic function which is bounded below (i.e., there exists a real number C such that $u(z) \geq C$ for all $z \in \mathbb{C}$), then u must be a constant.

End of Session 2

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2015

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. In this problem $(X, \mathcal{M}, \mu)$ denotes an arbitrary measure space.
A) State the monotone convergence theorem.
B) Prove that if $f_n : X \to [0, \infty)$ is measurable and non-negative for each positive integer n, then

$$\int_X (\liminf_{n\to\infty} f_n)\, d\mu \quad \leq \quad \liminf_{n\to\infty} \int_X f_n\, d\mu. \tag{1}$$

C) Give an example showing that (1) can be false if some of the functions f_n take negative values.

2. Let f and f_k, $k = 1, 2, \ldots$, be Lebesgue measurable functions on $[0, 1]$ such that $f_k \to f$ almost everywhere in $[0, 1]$. Suppose that $M := \sup_k \|f_k\|_\infty < \infty$. Show that for every $g \in L^p([0, 1])$, $1 \leq p \leq \infty$,

$$\lim_{k\to\infty} \int_0^1 f_k g\, dx \quad = \quad \int_0^1 f g\, dx\, .$$

3. Let (X, d_X) be a metric space and let (Y, d_Y) be a complete metric space. Let S be a dense subset of X and let $f : S \to Y$ be an L-Lipschitz map, meaning that $d_Y(f(x), f(y)) \leq L d_X(x, y)$ for all $x, y \in S$. Show there is a unique L-Lipschitz map $\bar{f} : X \to Y$ such that $\bar{f}|_S = f$ (that is, $\bar{f}$ is an extension of f).

Exam continues...

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let H be a Hilbert space endowed with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Recall that a sequence $\{u_k\}_{k\geq 1}$ in H is said to *converge weakly* to a limit $u \in H$ if for all $v \in H$, $\lim_{k\to\infty}(v, u_k - u) = 0$. In contrast, $\{u_k\}_{k\geq 1}$ *converges strongly* to $u \in H$ if $\lim_{k\to\infty} \|u_k - u\| = 0$.

Suppose that a sequence $\{u_k\}_{k\geq 1}$ in H converges weakly to $u \in H$, and that furthermore $\lim_{k\to\infty} \|u_k\|_2 = \|u\|_2$. Show that $\{u_k\}_{k\geq 1}$ in fact converges strongly to u.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n, g_n, h_n \in L^1(X, \mathcal{M}, \mu)$, $n \geq 1$, satisfy the inequalities

$$f_n(x) \leq g_n(x) \leq h_n(x) \,, \qquad \text{for all } n \geq 1 \text{ and } x \in X \,.$$

Let $f(x)$, $g(x)$, and $h(x)$ be functions such that

$$\lim_{n\to\infty} f_n(x) = f(x) \,, \quad \lim_{n\to\infty} g_n(x) = g(x) \,, \quad \text{and} \quad \lim_{n\to\infty} h_n(x) = h(x) \,,$$

for almost all $x \in X$. Furthermore assume that $f(x),\ h(x) \in L^1(X)$ with

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu \quad \text{and} \quad \lim_{n\to\infty} \int_X h_n(x)\,\mathrm{d}\mu = \int_X h(x)\,\mathrm{d}\mu \,.$$

Show that

$$\lim_{n\to\infty} \int_X g_n(x)\,\mathrm{d}\mu = \int_X g(x)\,\mathrm{d}\mu \,.$$

(HINT: Look at $h_n - g_n$ and $g_n - f_n$.)

End of Session 3