Rutgers - Graduate Program in Mathematics
Written Qualifying Examination

Spring 1996

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

First Day – Part I: Answer each of the following three questions.

1. Compute

$$\frac{1}{2\pi i}\int\limits_{|z|=100.5}\frac{\pi z}{\sin \pi z}dz.$$

2. Let the function $y = f(x)$ satisfy

$$\begin{cases} y'' = \lambda(x)y \text{ for } 0 \leq x \leq 1 \\ y(0) = y(1) = 0, \end{cases}$$

where $\lambda(x) > 0$ for $0 \leq x \leq 1$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

3. Let p be a prime integer. Prove that a group of order p^2 is Abelian.

First Day – Part II: Answer <u>three</u> out of the following six questions.

4. Suppose that $\{A_n \mid n = 1, 2, \ldots\}$ is a sequence of real $d \times d$ matrices such that $\lim(A_n)$ exists and equals A. Determine whether each of the following two statements is true or false. If the statement is true, prove it. If it is false, give a counterexample.

a) $\exists n_0, \forall n \geq n_0, \operatorname{rank}(A_n) \geq \operatorname{rank}(A)$;

b) $\exists n_0, \forall n \geq n_0, \operatorname{rank}(A_n) \leq \operatorname{rank}(A)$.

5. Let $f_n : [0, 1] \longrightarrow [0, \infty)$ be a Lebesgue integrable function for $n = 1, 2, \ldots$. Assume that

$$\int_0^1 f_n(x)dx = 1 \text{ and } \int_{1/n}^1 f_n(x)dx < 1/n$$

for all n. Let $g : [0, 1] \longrightarrow [0, \infty]$ be defined by

$$g(x) = \sup\{f_n(x) \mid n = 1, 2, 3, \ldots\}.$$

Prove that

$$\int_0^1 g(x)dx = \infty.$$

6. Write $E = \{z \in \mathbb{C} : |z| = 1\}$, $\partial E = \{z \in \mathbb{C} : |z| = 1\}$, and $\bar{E} = \partial E \cup E$. Let $f : \bar{E} \to \bar{E}$ be continuous and holomorphic on E. Assume that $f(\partial E)$ is contained in ∂E. Assume also that $f'(0) \neq 0$ and that $f(z) = 0 \Leftrightarrow z = 0$. Show that $f : E \to E$ is one-to-one and onto.

7. Let A be the 3-dimensional algebra over $\mathbb{C}$ with basis 1, x, x^2 and $x^3 = 1$. Let B be the 3-dimensional commutative algebra over $\mathbb{C}$ with basis 1, x, y and $x^2 = xy = y^2 = 0$. Show that A has only finitely many ideals, but that B has infinitely many.

8. Let X be a metric space with no isolated points. Assume that the set X is countably infinite. Show that X cannot be compact.

9. Let R be a commutative ring with unity 1, and let $f(x) = 1 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m$ be polynomials with coefficients a_i, $b_j \in R$. If $n > 0$ and $f(x)g(x) = 1$, show that $a_n^{r+1}b_{m-r} = 0$ for all r and that the elements $a_1, a_2, \ldots, a_n$ are nilpotent in R.

Second Day – Part I: Answer each of the following three questions.

1. Let f be a measurable function on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} (1+|x|)|f(x)|dx < \infty.$$

Show that

$$F(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} f(x)dx$$

is differentiable at each $\xi \in \mathbb{R}$.

2. The least upper bound axiom for the real numbers $\mathbb{R}$ states that every non-empty $S \subset \mathbb{R}$ which is bounded above has a least upper bound. Derive from this axiom, without citing any theorems, that the interval $[0, 1]$ is a compact subset of $\mathbb{R}$.

3. Let $e_1, e_2, \ldots, e_{n+2}$ be idempotent $n \times n$ matrices over a field. (A matrix e is idempotent if $e^2 = e$.) Show that at least two of these matrices are conjugate.

Second Day – Part II: Answer <u>three</u> of the following six questions.

4. Let M be a 6×6 real symmetric matrix whose minimal polynomial has degree 2. Let $Z(M)$ denote the linear subspace of $\mathbb{R}^{6\times 6}$ consisting of all symmetric matrices that commute with M. Determine the set of integers d that are possible values for the dimension of $Z(M)$.

5. Show that there is a branch of $\sqrt{z^2-1}$ which is holomorphic on $\mathbb{C} \setminus \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

6. Let H be a subgroup of the group $G = GL_3(\mathbb{F}_p)$ of invertible 3×3 matrices over the field $\mathbb{F}_p$ of order p. If H has order p^2, show that there is some $g \in G$ such that every matrix in gHg^{-1} is upper triangular.

7. A *plane graph* is a subset E of the plane $\mathbb{R}^2$ which can be written as a finite union of subspaces $e_1, e_2, \ldots, e_q$ (called *edges*), each of which is homeomorphic to the unit interval $[0, 1]$, such that distinct edges e_i and e_j intersect in at most one point, which is an endpoint of each edge. The set $V = \{v_1, v_2, \ldots, v_p\}$ of endpoints of edges is called the set of *vertices* of the graph. Show that the complement $\mathbb{R}^2 \setminus E$ has $c = q + 2 - p$ connected components.

8. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of the form $\mathbf{F}(\mathbf{r}) = g(|\mathbf{r}|^2)\mathbf{r}$ for $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$, where $g : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function. Let Σ be a smooth surface in $\mathbb{R}^3$ whose boundary is a simple closed curve γ. Show that the line integral

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0.$$

9. For $n = 1, 2, \ldots$ let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and assume that $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in \mathbb{R}$. Let $a \in \mathbb{R}$. Show that $\{x \in \mathbb{R} | f(x) > a\}$ is a countable union of closed sets.