Rutgers University - Graduate Program in Mathematics
Written Qualifying Examination

Spring 1997

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

First Day – Part I: Answer each of the following three questions.

1. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} \qquad (n = 0,\, 1,\, 2,\, \cdots)$$
using complex analysis.

2. Assume that $f(x)$ is differentiable and $f'(x)$ is strictly increasing for $x \geq 0$. If $f(0) = 0$, prove that $f(x)/x$ is strictly increasing for $x > 0$.

3. Let A be an $n \times n$ matrix with rational entries satisfying the equation $A^2 - 2I = 0$. Prove that n is even.

First Day – Part II: Answer three out of the following six questions.

4. Let $\Omega \subset \mathbf{C}$ be a bounded domain with smooth boundary. Let f be analytic in Ω and continuous up to the boundary. Assume that $|f(z)|$ is a constant for all z on the boundary of Ω. Prove that either f is identically equal to some constant in Ω, or $f(z) = 0$ has a solution in Ω.

5. Let G be a group with finitely many subgroups. Prove that G is finite.

6. Evaluate the sum of the infinite series

$$1^2/0! + 2^2/1! + 3^2/2! + \cdots + (n+1)^2/n! + \cdots$$

7. Let A, B, S be $n \times n$ complex matrices satisfying $AS = SB$ and $S \neq 0$. Prove that A and B have a common eigenvalue.

8. Let f be any Lebesgue measurable function on $\mathbf{R}^n$. Let $|E|$ denote the Lebesgue measure of a set E that is measurable. Prove that for $0 < p < \infty$ we have

$$p \int_0^\infty \lambda^{p-1} |\{x \,:\, |f(x)| > \lambda\}| \, d\lambda = \int_{\mathbf{R}^n} |f|^p \, dx.$$

9. Let $f \,:\, \mathbf{R}^n \to \mathbf{R}^n$ be a continuously differentiable function. Assume furthermore that the Jacobian of f is not zero at any point. Assume that f is proper (the inverse image of a compact set is compact). Prove that f is onto. Give an example in $\mathbf{R}^2$ to show that one cannot drop the assumption of being proper.

Second Day – Part I: Answer each of the following three questions.

1. We are given a 3×3 matrix A with real entries. Furthermore, we are given that A is not triangulable over the real numbers (not similar to an upper or lower triangular matrix). Prove that A is diagonalizable over the complex numbers.

2. Given that:

$$I = \int_0^\infty \frac{\sin x}{x}\, dx,$$

(a)Show that I is finite as an appropriate improper integral, by simple calculus.
(b) Compute I using complex analysis.

3. Let $u = u(x, y)$ be a smooth function defined on $\mathbf{R}^2$ which is 2π-periodic in each variable, i.e.,

$$u(x + 2n\pi,\ y + 2m\pi) = u(x, y) \qquad \text{for all } x, y \in \mathbf{R},\ \ n, m \in \mathbf{Z}.$$

Prove:

$$\int_0^{2\pi} \int_0^{2\pi} (u_{xx} u_{yy} - u_{xy}^2)\, dx dy = 0.$$

Second Day – Part II: Answer <u>three</u> of the following six questions.

4. Let $f(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$, where $c_i \neq 0$. Assume that the r_i are distinct. Prove that f cannot have n real zeros.

5. Let f be a holomorphic function in the unit disk $D = \{z \,:\, |z| < 1\}$, which is continuous on the closed disk. Suppose that f vanishes on the arc, given by the points$\{e^{i\theta} \,:\, 0 \leq \theta \leq \pi\}$. Show $f(z) = 0$ for all $z \in D$.

6. Let A and B be disjoint compact subsets of a Hausdorff topological space. Prove the existence of disjoint open sets U and V satisfying $A \subset U$ and $B \subset V$.

7. Let G be a group of order 90. Prove that G is solvable.

8. Let $f(x) = 2^{-n}$ if $x = k2^{-n}$ where k is odd and n is a natural number, and $f(x) = 0$ otherwise. Prove that f^2 has bounded variation on the interval $[0, 1]$.

9. Give an example of a measurable function $f \,:\, \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that the two double integrals

$$\int_{\mathbf{R}} \int_{\mathbf{R}} f(x,y)\, dxdy \quad \text{and} \quad \int_{\mathbf{R}} \int_{\mathbf{R}} f(x,y)\, dydx$$

are defined, but are not equal. Fubini's Theorem tells us that the condition $\int \int |f(x,y)|\, dxdy < \infty$ cannot be satisfied by such a function.