Rutgers University - Graduate Program in Mathematics

Written Qualifying Examination

Spring 1999

Day 1

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

First Day – Part I: Answer each of the following three questions.

Question 1. (a) Prove that if $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = x^2 \sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$, then f is of bounded variation on $[0,1]$ and the total variation satisfies $V(f;[0,1]) \leq 2$.

(b) Suppose that $g : [0,1] \to \mathbb{R}$ is defined by $g(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $g(0) = 0$. Prove or disprove that g is of bounded variation on $[0,1]$.

In parts (a) and (b), you must explain carefully which theorem(s) you are using.

Question 2. Prove that there is no simple group of order 1960.

Question 3. Find the order of the pole of

$$f(z) = \frac{1}{(6\sin z + z^3 - 6z)^2}$$

at $z = 0$.

First Day – Part II: Answer three out of the following six questions.

Question 4. Suppose that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with finite Lebesgue measure $|E| > 0$. For each real number $p \geq 1$ and Lebesgue measurable function $f : E \to \mathbb{R}$, define

$$N_p(f) = \left(\frac{1}{|E|} \int_E |f(x)|^p \, dx \right)^{1/p}.$$

Prove that if $1 \leq p < q$, then $N_p(f) \leq N_q(f)$.

Question 5. Prove that if G is a nontrivial finite group, then the following two statements are equivalent.

(a) G is an elementary abelian p-group for some prime p; ie. G is isomorphic to a direct product of cyclic groups of order p.

(b) The automorphism group $\mathrm{Aut}(G)$ of G acts transitively on the set $G \smallsetminus \{1\}$ of nonidentity elements of G.

Question 6. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a series such that $a_n \in \mathbb{Z}$ for each $n = 0, 1, 2, \ldots$ Prove that if the radius of convergence R of $f(z)$ is greater than 1, then $f(z)$ is a polynomial.

Question 7. If D is a subset of $\mathbb{R}^n$, then a map $f : D \to \mathbb{R}^n$ is called *Lipschitzian* if there exists a constant C such that $||f(x) - f(y)|| \leq C||x - y||$ for all $x, y \in D$. Prove that if D is a Lebesgue measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is Lipschitzian, then $f(D) = \{f(x) \mid x \in D\}$ is Lebesgue measurable. (*Hint.* First prove that sets of measure zero are mapped to sets of measure zero. Is the result obvious for compact sets?)

Question 8. For each $n \geq 2$, let

$$f_n(x) = x^{n-1} + x^{n-2} + \cdots + x + 1 \in \mathbb{Q}[x].$$

Prove that $f_n(x)$ is irreducible over $\mathbb{Q}$ iff n is a prime. (*Hint.* To show that $f_n(x)$ is irreducible over $\mathbb{Q}$, it is enough to show that $f_n(x+1)$ is irreducible over $\mathbb{Q}$.)

Question 9. Prove that if an analytic function $f(z)$ on the extended complex plane $\mathbb{C} \cup \{\infty\}$ has only one first order pole, then it must have the form

$$f(z) = \frac{az + b}{cz + d},$$

for some $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

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Day 2

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

Second Day – Part I: Answer each of the following three questions.

Question 1. Calculate the following integral:

$$\int_C \frac{zdz}{(z-1)(z-2)^2},$$

where C is the curve given by $|z-2| = \frac{1}{2}$.

Question 2. Suppose that A, B are $n \times n$ complex matrices, each with no eigenvalues of multiplicity 4 or more. Show that if A and B have the same characteristic polynomial and the same minimal polynomial, then A and B are similar.

Question 3. (a) Describe carefully the construction of the Cantor set $C \subset [0,1]$.

(b) Let δ be a fixed number such that $0 < \delta < 1$. Construct a subset $D \subset [0,1]$ in the same manner as the Cantor set C, except that at the k^{th} stage of the construction, each of the removed open intervals has length $\delta 3^{-k}$. Prove that the resulting generalized Cantor set D is compact, has Lebesgue measure $1-\delta$ and contains no intervals of positive length.

Second Day – Part II: Answer <u>three</u> of the following six questions.

Question 4. Let $f(z)$ be a nonconstant analytic function on a bounded closed region D. Prove that if $f(z)$ is constant on the boundary of D, then $f(z)$ has a zero inside D.

Question 5. Let A be a finite-dimensional commutative algebra over a field F. Show that every prime ideal P of A is maximal.

Question 6. Suppose that $E \subset \mathbb{R}$ is Lebesgue measurable with finite Lebesgue measure $|E|$ and that $f : E \to \mathbb{R}$ is a Lebesgue measurable function. Prove that

$$||f||_\infty = \lim_{p\to\infty} ||f||_p.$$

Question 7. Show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x+i} dx = \frac{\pi}{e}.$$

(Hint: $x^2 + 1 = (x+i)(x-i)$.)

Question 8. Let S_n be the symmetric group on the set $X = \{1, \dots, n\}$ and let G be a subgroup of S_n. Define a binary relation $\sim$ on X by $a \sim b$ iff either $a = b$ or the transposition $(\,a\,b\,)$ is in G.

(a) Verify that $\sim$ is an equivalence relation on X.

(b) Show that if $n = p$ is a prime and G contains both a p-cycle and a transposition, then $G = S_p$.

Question 9. (a) If $f \in L^2(0, 2\pi)$, show that

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx\, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx\, dx = 0.$$

(b) Use part (a) to prove that if $f \in L^1(0, 2\pi)$, then

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx\, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx\, dx = 0.$$