RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2014, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

Before starting your exam,

- You will be given an anonymizer (a secret ID) to be used on your answer books (so that your real identify is not revealed to the grading committee). Be sure your secret ID, not your real name, is on each book you are submitting. The same secret ID will be used for both days, so please keep it safe.
- Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).
- If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam,

- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Prove that up to isomorphisms, there are exactly two non-isomorphic groups of order 21. Describe these groups using either generators and relations, or an appropriate description in terms of products. Indicate if these groups are abelian/non–abelian.

2. Let f be a twice continuously differentiable function on an open set $U \subset \mathbb{R}^3$ For $r > 0$, let $B_r(\mathbf{x})$ denote the open ball of radius r centered at $\mathbf{x}$, and let $S_r(\mathbf{x})$ be the sphere of radius r centered at $\mathbf{x}$. For each fixed $\mathbf{x} \in U$ and each $R > 0$ such that $B_R(\mathbf{x}) \subset U$, define the function $A(r)$ for $0 \le r < R$ by

$$A(r) = \frac{1}{4\pi} \int_{S^2} f(\mathbf{x} + r\mathbf{n}) \, d\sigma(\mathbf{n}) \, ,$$

where σ denotes surface measure on the unit sphere S^2 (the set of all vectors $\mathbf{n}$ in $\mathbb{R}^3$ with $\|\mathbf{n}\| = 1$).

(a) Show that $A(r)$ is differentiable on $(0, R)$ and that

$$A'(r) = \frac{1}{4\pi} \int_{S^2} \nabla f(\mathbf{x} + r\mathbf{n}) \cdot \mathbf{n} \, d\sigma(\mathbf{n}) \quad \text{for all } 0 < r < R,$$

where ∇ is the gradient operator.

(b) Show that if $\Delta f(\mathbf{y}) = 0$ for all $\mathbf{y} \in U$, then

$$A(r) = f(\mathbf{x}) \quad \text{for } 0 < r < R.$$

Here Δ denotes the *Laplace operator*; $\Delta f = \nabla \cdot \nabla f$.

3. Let $f(z)$ be an analytic function defined on some open neighborhood of the closed disk $|z| \le r$. Assume that there exists $a > 0$ such that $|f(0)| < a$ and $|f(z)| > a$ for z on the boundary circle $|z| = r$. Prove that $f(z)$ must have a zero in the open disk $|z| < r$.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let m denote Lebesgue measure on $[0,1]$. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$ such that

$$\int_{[0,1]} |f_n(x)|^2 \, dm(x) \leq 1 \quad \text{for all } n.$$

Suppose that the sequence $\{f_n\}$ converges to zero in measure. Show that

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) \, dm(x) = 0 \, .$$

5. Let the group $\mathrm{GL}(2,\mathbb{R})$ of invertible 2×2 real matrices act on the set SYM of 2×2 real symmetric matrices by $S \mapsto A^T S A$ for $S \in \mathrm{SYM}$ and $A \in \mathrm{GL}(2,\mathbb{R})$.

(a) Show that each orbit has a representative which is a diagonal matrix with entries in the set $\{-1, 0, 1\}$.

(b) Find a set of representatives for each orbit under this action.

6. Suppose that $f(z)$ is an entire function on $\mathbb{C}$, and that there exist constants $M > 0$ and $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} e^{2\pi M|y|} \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that

$$\int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx = 0 \quad \text{for all } \xi \in \mathbb{R} \text{ with } |\xi| > M \, .$$

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \in \mathcal{M}$ is said to be an *atom* for μ if $0 < \mu(E)$ and whenever $F \in \mathcal{M}$ and $F \subset E$, either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. A measure space in which there are no atoms is said to be *diffuse*. Suppose $(X, \mathcal{M}, \mu)$ is a diffuse measure space.

(a) Show that for each $n \in \mathbb{N}$ and each $E \in \mathcal{M}$ with $\mu(E) > 0$, there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < 1/n$.

(b) Let $E \in \mathcal{M}$ satisfy $0 < \mu(E) < \infty$. Show that for each $a \in (0,1)$,

$$\sup\{\mu(F) \; : \; F \in \mathcal{M}, \quad F \subset E, \quad \mu(F) \leq a\mu(E)\} = a\mu(E) \, .$$

8. Determine the Jordan canonical form J of the matrix

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Show the details of your reasoning and your calculation.

9. Suppose that $f(z)$ is a meromorphic function on the extended complex plane (including $z = \infty$) with only two poles: $z = -1$ is a pole of order 1 with $\dfrac{1}{z+1}$ as the principal part, and $z = 2$ is a pole of order 3 with $\dfrac{2}{z-2} + \dfrac{4}{(z-2)^3}$ as the principal part. Suppose further that $f(0) = 1$.

(a) Determine $\int_{|z|=4} f(z)\,dz$.

(b) Determine the number of solutions to $f(z) = 1$ in the extended complex plane.

(c) Determine $f(z)$ explicitly.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2014, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts.

Before starting your exam, note the following.

- You have been given an anonymizer (a secret ID) on Day 1 to be used on your answer books (so that your real identify is not revealed to the grading committee). Be sure your secret ID, not your real name, is on each book you are submitting.
- Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).
- If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam, be sure to do the following.

- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Let m denote Lebesgue measure on $[0,1]$. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$ with values in $[0,\infty]$ such that

$$\lim_{n\to\infty} f_n(x) = 0$$

almost everywhere and

$$\int_{[0,1]} f_n(x)\,\mathrm{d}m(x) = 1 \quad \text{for all } n.$$

Define $g(x) = \sup_{n\in\mathbb{N}}\{f_n(x)\}$. Show that

$$\int_{[0,1]} g(x)\,\mathrm{d}m(x) = \infty\,.$$

2. Let A be a free abelian group and $a \in A$ a nonzero element.

(a) Show that there are only finitely many integers n and elements $x \in A$ such that $a = nx$.

(b) Show (using part (a) or via other proof) that the group $\mathbb{Q}$ of rational numbers under addition is not a free abelian group.

3. Calculate the definite integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+9)}\,\mathrm{d}x$$

using residues and contour integrals. Justify any arguments involving limits that you use in your calculation.

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let (X, ρ) be a metric space.

(a) Suppose that (X, ρ) is separable. Show that if Y is any subset of X, then (Y, ρ) is separable.

(b) Suppose that (X, ρ) is compact. Show that (X, ρ) is separable.

5. Let p and q be distinct primes. Prove that there is no simple group of order pq.

6. Let m denote Lebesgue measure on $[0, 1]$. Let $0 \leq a < b \leq 1$. Does there exist an open set $U \subset [0, 1]$ such that $m(U) = a$, but $m(\overline{U}) = b$? (Here $\overline{U}$ denotes the closure of U.) Prove your answer is correct.

7. Let C_1 and C_2 be simple closed curves in $\mathbb{C}$ and assume that C_2 is in the interior of C_1. Let U be the (open) region bounded by C_1 and C_2.

(a) Prove that an analytic function $f(z)$ on U can be decomposed as

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is analytic in the interior of C_1 and $f_2(z)$ is analytic in the exterior of C_2 (including ∞).

(b) Prove that the decomposition in (a) is unique up to an additive constant; that is, prove that if

$$f(z) = f_1(z) + f_2(z) = g_1(z) + g_2(z),$$

where $f_1(z)$ and $g_1(z)$ are analytic in the interior of C_1 and $f_2(z)$ and $g_2(z)$ are analytic in the exterior of C_2 (including ∞), then there exists a constant A such that

$$f_1(z) = g_1(z) + A \quad \text{and} \quad f_2(z) = g_2(z) - A\,.$$

8. Suppose that f is holomorphic in a neighborhood of z_0, and that all complex derivatives of f up to order $m-1$ at z_0 vanish, namely, $f^{(j)}(z_0) = 0$ for all $j = 0, \ldots, m-1$, but that $f^{(m)}(z_0) \neq 0$.

(a) Prove that there exist $\epsilon > 0$ and $\delta > 0$ such that, for every $k \in \mathbb{N}$, the equation

$$G_k(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \epsilon} \frac{\zeta^k f'(\zeta)}{f(\zeta) - w} \, d\zeta$$

defines a holomorphic function of w in the set

$$D_\delta(f(z_0)) = \{w \in \mathbb{C} : |w - f(z_0)| < \delta\}.$$

(b) Prove that, in the context of (a), if $w \in D_\delta(f(z_0))$ then the equation $f(z) - w = 0$ has m roots (counted with multiplicity), $z_1, \ldots, z_m$, inside $|z - z_0| < \epsilon$, and that

$$G_k(w) = \sum_{j=1}^{m} z_j^k.$$

9. Let $\mathbb{C}[x,y]$ be the ring of polynomials in two variables x, y with complex coefficients. Let $\mathcal{J}$ be the ideal in $\mathbb{C}[x,y]$ generated by the polynomial $y^2 - x^3$. Let $\mathcal{R}$ be the quotient ring $\mathbb{C}[x,y]/\mathcal{J}$.

(a) Show that $\mathcal{R}$ is an integral domain.

(b) Prove or disprove: The ring $\mathcal{R}$ is a principal ideal domain.

Exam Day 2 End