RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 13, 2011, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Let f be a complex valued measurable function on $\mathbb{R}$. Let μ be the Lebesgue measure and suppose that for each $a < b$,

$$\left| \int_a^b f d\mu \right| \leq b - a.$$

Prove that $|f(x)| \leq 1$ for almost every x.

2. Use contour integration to evaluate

$$\int_0^\infty \frac{1}{(1+x^2)^2} dx.$$

Be clear about any computation of residues and about any computations of limits of integrals.

3. Let S_9 denote the symmetric group on $\{1, 2, ..., 9\}$ and let $\sigma \in S_9$ be given (in table form) by

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 8 & 1 & 7 & 2 & 6 & 3 & 4 \end{bmatrix}.$$

As usual $C(\sigma)$, the centralizer of σ in S_9, is defined to be $C(\sigma) = \{\tau \in S_9 | \tau\sigma = \sigma\tau\}$. Find $|C(\sigma)|$ and justify your answer.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Recall that a set X in a topological space is called a G_δ set when it is a countable intersection of open sets, and it is called an F_σ set when it is a countable union of closed sets. Let μ denote Lebesgue measure on $\mathbb{R}$. Show that for every Borel set $A \subset \mathbb{R}$ there is a G_δ set G and an F_σ set F such that

$$F \subset A \subset G, \qquad \mu(G \cap F^c) = 0.$$

Here $F^c = \mathbb{R} \setminus F$.

5. Let f be analytic on the unit disc D, and assume that $|f(z)| < 1$ for all $z \in D$. Prove that if there exist two distinct points a and b in the disc which are fixed points, that is, $f(a) = a$ and $f(b) = b$ then $f(z) = z$ for all $z \in D$.

6. Prove that there exists no simple group of order 80.

7. Let X and Y be topological spaces and $f : X \to Y$ and $g : X \to Y$ be continuous functions. Prove that if Y is a Hausdorff space then $\{x \in X : f(x) = g(x)\}$ is closed.

8. Let (f_n) be a sequence of nonnegative integrable functions on $[0, 1]$ converging almost everywhere to a function $f(x)$. Prove that if

$$\lim_{n\to\infty} \int_{[0,1]} f_n d\mu = \int_{[0,1]} f d\mu$$

then

$$\lim_{n\to\infty} f_n = f$$

in $L^1[0, 1]$.

The exam continues on next page

9. Let A and B be commuting 8 by 8 diagonalizable matrices over the real numbers with characteristic polynomials

$$\det(A - \lambda I) = (\lambda - 1)^3(\lambda - 3)^5$$

and

$$\det(B - \lambda I) = \lambda^2(\lambda - 4)^6.$$

Suppose the minimum polynomial of $A - B$ is

$$(\lambda^2 - 1)(\lambda^2 - 9).$$

Find the dimension of the vector space of all 8 by 8 real matrices that commute with both A and B.

Day 1 Exam Ends

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 14, 2011, Day 2

This examination is given in two three-hour sessions, today's being the second part.
At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Let $f(x)$ be a function on $[0, 1]$ and suppose that $f'(x)$ is defined for all $0 \leq x \leq 1$. Prove that $f'(x)$ is a measurable function.

2. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie in $\{1 \leq |z| \leq 2\}$.

3. Are the quotient rings $\mathbb{Z}[x]/(x^3+1)$ and $\mathbb{Z}[x]/(x^3+2x^2+x+1)$ isomorphic? Provide full justification for your answer.

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Find the Laurent expansion of $f(z) = (1 - z^2)e^{\frac{1}{z}}$ around $z = 0$. Compute the residue of $f(z)$ at 0.

5. Let f be a complex valued measurable function on $\mathbb{R}$. Let μ be Lebesgue measure and suppose that for each $g \in L^2(\mu)$, $fg \in L^1(\mu)$. Show that $f \in L^2(\mu)$.

6. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be vectors over the field $F = \mathbf{Z}/3$. Show that the bilinear forms $B(x, y) = -x_1y_1 - x_2y_2$ and $D(x, y) = x_1y_1 + x_2y_2$ are equivalent.

7. Consider the curve $S = \{(x, \sin(1/x)) : x \in (0, 1]\} \subseteq \mathbb{R}^2$. Let $T = S \cup (\{0\} \times [-1, 1])$. Show that T is a connected subset of $\mathbb{R}^2$.

8. Let G be a finite group. Prove that G is cyclic if and only if G has exactly one subgroup of order n for each positive integer n dividing $|G|$.

9. Exhibit a conformal map $f : U \to D$, (that is, a bijective map f from U to D, such that both f and its inverse are holomorphic), where D is the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and U is the set $\{z \in D : \text{Re } z > 0\}$.

Exam Day 2 Ends