# A COUNTEREXAMPLE TO THE SECOND INEQUALITY OF COROLLARY (19.10)IN THE MONOGRAPH "RICCI FLOW AND THE POINCARE CONJECTURE" BY J.MORGAN AND G.TIAN

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## 1.Introduction.

We provide in the sequel a counterexample to the second inequality of Corollary (19.10) of [1], in line with what was announced in [2].

## 2. The counterexample.

Observe that the inequality implies that a curve which is a geodesic for a given metric remains a geodesic for the metric evolved through the Ricci flow as the curve itself is evolved through the curve shortening flow $H$.

However, the norm of $H$ is $k$ and therefore the curve itself does not move under the curve shortening flow.

It follows that, if the inequality holds, a curve that is a geodesic remains, without moving, a geodesic for the evolved metric through the Ricci flow. We provide below a counterexample to this conclusion.

The equation of a geodesic reads:

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$

We introduce coordinates along a small piece of curve of this geodesic so that this small piece of curve defines the $x^1$-axis of coordinates. It follows that, for this set of coordinates, on this small piece of curve:

$$\dot{x}^1(t) \gneq 0$$

, whereas $\dot{x}^s(t) = 0, s \neq 1$

Then, on this small piece of curve, the geodesic equation verified for $s \neq 1$ reads:

$$\ddot{x}^s + \Gamma^s_{11}\dot{x}^1\dot{x}^1 = 0 \Longleftrightarrow \Gamma^s_{11}\dot{x}^1\dot{x}^1 = 0$$

As the metric evolves through the Ricci flow, $\ddot{x}^s$ remains equal to zero, for $s \neq 1$, on the piece of curve. $\Gamma^s_{11}$ was zero at the time zero of the evolution. It follows that the first variation of $\Gamma^s_{11}$, which we denote $\delta\Gamma^s_{11}$ should also be zero along this piece of curve.

We choose an arbitrary point $x_0$ on this piece of curve. We may assume that, at this point, the metric tensor reads $g_{ij}(x_0) = \delta_{ij}$. There is no loss of generality in this requirement.

Then,

$$\Gamma^s_{11}(x_0) = 1/2(2\frac{\partial g_{1s}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^s})(x_0)$$

The formula for $\delta\Gamma^s_{11}$ reads (use geodesic normal coordinates in order to derive the formula. $\delta\Gamma^k_{ij}$ are components of a tensor. Therefore, the computation in any set of coordinates, eg geodesic normal coordinates, provides the formula for another set of coordinates):

$$\delta\Gamma^s_{11}(x_0) = 1/2(2\nabla_1\delta g_{1s} - \nabla_s\delta g_{11})(x_0)$$

where we used the fact that $g_{ij}(x_0) = \delta_{ij}$.

We know that $\delta g_{ij} = -2R_{ij}$, so that:

$$\delta\Gamma^s_{11}(x_0) = (\nabla_s R_{11} - 2\nabla_1 R_{1s})(x_0) = 0$$

We have:

$$R_{11} = \frac{\partial \Gamma^t_{11}}{\partial x^t} - \frac{\partial \Gamma^t_{1t}}{\partial x^1} + O(\Gamma^2)$$

$$R_{1s} = \frac{\partial \Gamma^t_{1s}}{\partial x^t} - \frac{\partial \Gamma^t_{1t}}{\partial x^s} + O(\Gamma^2)$$

The expression of the covariant derivative of $R_{11}$ and $R_{1s}$ is complicated; but there are third derivatives of the metric tensor in $\delta\Gamma^s_{11}(x_0)$. They are derived as if we were computing in a geodesic normal coordinates system. They are the third derivatives in:

$$\frac{\partial}{\partial x^s}(\frac{\partial \Gamma^t_{11}}{\partial x^t} - \frac{\partial \Gamma^t_{1t}}{\partial x^1})(x_0) - 2\frac{\partial}{\partial x^1}(\frac{\partial \Gamma^t_{1s}}{\partial x^t} - \frac{\partial \Gamma^t_{1t}}{\partial x^s})(x_0)$$

Computing, we find:

$$-(2\frac{\partial^3 g_{st}}{\partial {x^1}^2 \partial x^t} - 2\frac{\partial^3 g_{1s}}{\partial {x^t}^2 \partial x^1} + \frac{\partial^3 g_{11}}{\partial {x^t}^2 \partial x^1} - \frac{\partial^3 g_{tt}}{\partial {x^1}^2 \partial x^s})(x_0)$$

For $t \neq s, t \neq 1$, we claim that $(\frac{\partial^3 g_{tt}}{\partial {x^1}^2 \partial x^s})(x_0)$, for $t \neq s, s \neq 1$, is a free parameter along this piece of curve: indeed, the only coordinate that is non-zero along this piece of curve is $x^1$. Thus, all $\dot{x}^r, r \neq 1$ are zero and the geodesic equation involves only the Christoffel symbols $\Gamma^m_{11}$. $g_{tt}$, $t \neq 1, t \neq s$, can interfere which this equation, but only through the coefficients $g^{mq}$ of the inverse of the metric tensor in front of the Christoffel symbols. These $g^{mq}$s can be subject to constraints along the piece of curve. But, no derivative of these $g^{mq}$, hence no derivative of $g_{tt}$, $t$ as above, is involved in the equations that are verified. All the terms with derivatives of the metric in these Christoffel symbols involve $g_{1q}$, ie one of the indexes is 1.

Thus no transversal derivative for $g_{tt}$, $t$ as above, is involved in the geodesic equation. The behavior of $g_{tt}$ and its derivatives along $x^1$ transversally to the piece of curve is completely free. The conclusion follows

The other terms involve only products and powers of the first and second derivatives of the metric tensor, on this piece of curve. Also, on this piece of curve, only $\partial/\partial x^1$ is a derivative along the curve. The other derivatives $\partial/\partial x^t, t \neq 1$ are transverse to this piece of curve.

$(\frac{\partial^3 g_{tt}}{\partial {x^1}^2 \partial x^s})(x_0), t \neq 1, t \neq s$ can be taken non-zero and large as we please in a small neighborhood of the curve, so that $\delta\Gamma^s_{11}(x_0)$ is non-zero, a contradiction.

# References

1. J.Morgan and G.Tian, *Ricci Flow and the Poincare Conjecture*, vol. 3, Clay Mathematics Monograph, AMS and Clay Institute, 2007.
2. A.Bahri, *Five gaps in Mathematics*, Advanced Non-linear Studies **Vol. 15, No. 2** (2015), 289-320.