Axioms for the Real Number System

General introduction.

The real number system is composed of a set $\mathbb{R}$, a distinguished subset $\mathbb{P}$, and two binary operations $+$ and $\times$. We use the notations $\mathbf{R}$ and $\mathbb{R}$ for both the set and the system, despite the ambiguity. When we use the term *number* we mean a real number. If we want to refer to any other number system we have to say so explicitly. The set $\mathbb{P}$ contains the numbers we want to distinguish as positive numbers. The binary operation $+$ will be the familiar operation of addition. The operation $\times$ is the familiar operation of multiplication. We now restate these introductory ideas as axioms.

G.1. $\mathbb{R}$ is a set.
G.2. There is a distinguished subset of $\mathbb{R}$ called $\mathbb{P}$, the set of "positive" numbers.
G.3. There are two binary operations, $+$ and $\times$ on $\mathbb{R}$.

Recall that a binary operation on $\mathbb{R}$ is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$.

About Addition

A.1. For all x and y in $\mathbb{R}$, $x + y = y + x$.
A.2. For all x, y, and w in $\mathbb{R}$, $(x + y) + w = x + (y + w)$.
A.3. There is a real number z, such that for all real x, $x + z = x$.
Note: We will show that there is only one such z. It is called "zero" and denoted 0.
A.4. For each real number x, there is a real number i, such that $x + i = 0$.
Note: For each real x, there is only one such object i.
It is called the "additive inverse of x " and is denoted by $-x$.

About Multiplication

M.1. For all x and y in $\mathbb{R}$, $x \times y = y \times x$.
M.2. For all x, y, and w in $\mathbb{R}$, $(x \times y) \times w = x \times (y \times w)$.
M.3. There is a real number u, such that $u \neq 0$ and for all real x, $x \times u = x$.
Note: There is exactly one such u. It is called "one" and is denoted by 1.
M.4. For each x in $\mathbb{R} - \{0\}$, there is a real number r such that $x \times r = 1$.
Note: For each non-zero x, there is exactly one such r.
It is called the "multiplicative inverse of x" and is denoted by $1/x$.

Connecting Addition and Multiplication

D.1. For all a, b, and c in $\mathbb{R}$, $(a + b) \times c = (a \times c) + (b \times c)$.

About the set $\mathbb{P}$.

These axioms allow us to define the usual order on $\mathbb{R}$ and to deduce the algebraic properties of order which are crucial to analysis.

O.1. For all x and y in $\mathbb{P}$, $x + y \in \mathbb{P}$ and $x \times y \in \mathbb{P}$.
O.2. For each real x, exactly one of the following three statements is true:

$$x \in \mathbb{P} \qquad\qquad x = 0 \qquad\qquad -x \in \mathbb{P}$$

Any system $\langle \mathbb{F}, +, \times \rangle$ satisfying the properties **A**, **M**, and **D** is a field. If the system also has a distinguished set of positive elements $\mathbb{P}$ which satisfies **O**, then the system is an ordered field. The real number system has one more distinctive property, completeness. It takes several definitions to build up to the concept of completeness.

Definitions:

1. *Subtraction* is a binary operation on $\mathbb{R}$, defined for all real x and y by

$$x - y = x + (-y).$$

2. *Division* is defined on all pairs (x, y) of reals with $y \neq 0$ by

$$x \div y = x/y = \frac{x}{y} = x \times (1/y)\,.$$

3. We define four *order relations* on $\mathbb{R}$ as follows. For all x and y in $\mathbb{R}$,

$$\begin{array}{ll} x < y \iff \{y + (-x) \in \mathbb{P}\} & x > y \iff y < x \iff \{x + (-y) \in \mathbb{P}\} \\ x \leq y \iff \{[x < y] \text{ or } [x = y]\} & x \geq y \iff y \leq x \iff \{[x > y] \text{ or } [x = y]\}. \end{array}$$

Note that with these definitions, $\mathbb{P} = \{x : 0 < x\} = \{x : x > 0\}\,.$

4. Suppose that $S \subseteq \mathbb{R}$ and that u and ℓ are real numbers. We introduce the following definitions and notations:

u is an *upper bound for S* iff for all s in S, $s \leq u$.

ℓ is a *lower bound* for S iff for all s in S, $\ell \leq s$.

$$\begin{array}{rcl} \mathcal{UB}(S) & = & \{u : u \text{ is an upper bound for } S.\} \\ \mathcal{LB}(S) & = & \{\ell : \ell \text{ is a lower bound for } S.\} \end{array}$$

S is *bounded above* iff $\mathcal{UB}(S) \neq \phi$

S is *bounded below* iff $\mathcal{LB}(S) \neq \phi$.

5. Suppose that $S \subseteq \mathbb{R}$ and $m \in \mathbb{R}$. We say that
   m is a *smallest element* in S iff $m \in S$ and, for all s in S ,$m \leq s$
   and
   m is a *largest element* in S iff $m \in S$ and for all s in S, $m \geq s$.

Notes:
(1) We will see that if S has a smallest element, then that element is unique.
(2) In such a case we denote that smallest element by $\min(S)$ for *minimum of S*.
(3) Similarly we will see that if S has a largest element, that element is unique and
(4) we will denoted this largest element by $\max(S)$ for *maximum* of S.

The completeness axiom

C1. For every subset S of $\mathbb{R}$,
if S is not empty and $\mathcal{UB}(S)$ is not empty, then $\mathcal{UB}(S)$ contains a smallest element.

Such a smallest element of a set of upper bounds for a set S, is called a *least upper bound* or a *supremum* for S. We will see that if a set has a least upper bound, then that least upper bound is unique and we will denote it by *lub*(S) or $\sup(S)$.

An ordered field $\langle \mathbb{F}, +, \times, \mathbb{P} \rangle$ satisfying the completeness axiom is called a complete ordered field. The real numbers form a complete ordered field. The rational numbers are an ordered field under the usual ordering, but – as we will see – do not satisfy the completeness axiom.