Homework, Math 311:03, Fall 2009

Sample Solutions to Various Problems from Chapter 0

0.1#6 TASK Suppose that A, B, C are sets.

(a) Prove that

$$\text{If } A \subseteq B, \text{ then } C - B \subseteq C - A.$$

(b) Either prove the converse or provide a counterexample.

PROOF

(a) Assume that $A \subseteq B$. [Note: Our book uses $\subset$ for the weak inclusion. I am using $\subseteq$ to emphasize that we are not limiting ourselves to the strong inclusion, the one that rules out equality.] I must show that

$$\text{for all } x,\ x \in C - B \Longrightarrow x \in C - A.$$

So consider an arbitrary x and assume $x \in C - B$. This means $x \in C$ and $x \notin B$. Since $A \subseteq B$ and $x \notin B$, we learn that $x \notin A$. Thus $x \in C - A$. $\square$

(b) The converse says

$$\text{If } C - B \subseteq C - A, \text{ then } A \subseteq B.$$

This need not be true. Consider the example

$$A = \{1\}\,,\ B = C = \phi$$

Then we have

$$C - A = \phi \text{ and } C - B = \phi, \text{ so } \ C - B \subseteq C - A \ \text{ but } A \subsetneq B$$

0.1#10 TASK b. Give a concise description of the set $B \doteqdot \bigcup_{n=1}^{\infty} (-n,n)$.

RESULT $B = \mathbb{R}$

REASONING By definition

$$B = \{x : \exists n \text{ in } \mathbb{N},\ x \in (-n,n)\}$$

B is a union of subsets of $\mathbb{R}$, so B is a subset of $\mathbb{R}$. To show the reverse inclusion, consider an arbitrary real r. We can find a positive integer n_r such that $-n_r < r < n_r$. So r belongs to the interval $(-n_r, n_r)$. Thus there is an index n, namely $n = n_r$, such that $r \in (-n,n)$. This shows that $r \in B$. $\square$

TASK d. Give a concise description of the set $D \doteqdot \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}\ ,\ 2+\frac{1}{n}\right)$.

RESULT

$$D = \left(-\frac{1}{1}\ ,\ 2+\frac{1}{1}\right) = (-1,3)$$

REASONING By definition

$$D = \left\{x\ :\ \exists n,\ x \in \left(-\frac{1}{n}\ ,\ 2+\frac{1}{n}\right)\right\}.$$

Suppose that $x \in D$. Then we can and do pick a positive integer, call it n_o,

$$x \in \left(-\frac{1}{n_o}\ ,\ 2+\frac{1}{n_0}\right)$$

Since $n_o \geq 1$, we get

$$\frac{1}{n_o} \leq 1 \quad \text{so} \quad -1 \leq -\frac{1}{n_o} < x < 2+\frac{1}{n_o} < 2+1 \quad \text{and } x \in (-1,3)\,.$$

This shows that

$$D \subseteq (-1,3)\,.$$

Now suppose that $x \in (-1,3)$. Then with $n = n_1 = 1$ we get

$$x \in (-1,3) = \left(-\frac{1}{n_1}\ ,\ 2+\frac{1}{n_1}\right) \quad \text{so} \quad x \in D.$$

0.3 #20 TASK Prove that

$$\forall n \text{ in } \mathbb{N}, \quad 1 + 3 + ... + (2n - 1) = n^2$$

PROOF This calls for a proof by induction. For each positive integer n, let $P(n)$ denote the assertion

$$\text{The sum of the first } n \text{ odd integers is } n^2$$

Base Step. Consider $n = 1$. The sum of the first n odd integers is just 1. Also $n^2 = 1$. So the assertion $P(1)$ is true.

Induction Step. Suppose that n is an arbitrary positive integer and that $P(n)$ is true. We must deduce the truth of $P(n + 1)$.

The sum of the first $n+1$ odd integers is the sum of the first n odd integers plus the $(n+1)^{st}$. This sum is, by the induction hypothesis just $n^2+(2n + 1)$. But

$$n^2 + (2n + 1) = (n + 1)^2$$

and this gives us

$$\text{The sum of the first } n + 1 \text{ odd integers is } (n + 1)^2$$

which is exactly $P(n + 1)$.

0.3 #24 TASK: Define a function f from $\mathbb{N}$ into $\mathbb{N}$ by

$$\begin{aligned} f(1) &= 1 \qquad f(2) = 2 \qquad f(3) = 3 \qquad \text{and} \\ \text{whenever } n &\geq 4,\ f(n) = f(n-1) + f(n-2) + f(n-3). \end{aligned}$$

Show that

$$\text{for all } n \text{ in } \mathbb{N},\ f(n) \leq 2^n$$

EXPLORATION We check the assertion for several values of n

$$\begin{aligned} \text{when } n &= 1,\ f(n) = f(1) = 1 \leq 2 = 2^1 = 2^n \\ \text{when } n &= 2,\ f(n) = f(2) = 2 \leq 4 = 2^2 = 2^n \\ \text{when } n &= 3,\ f(n) = f(3) = 3 \leq 8 = 2^3 = 2^n \\ \text{when } n &= 4,\ f(n) = f(4) = f(3) + f(2) + f(1) = 3 + 2 + 1 \leq 16 = 2^4 = 2^n \\ \text{when } n &= 5,\ f(n) = f(5) = f(4) + f(3) + f(2) = 6 + 3 + 2 \leq 32 = 2^5 = 2^n \end{aligned}$$

PROOF

For each n in $\mathbb{N}$, let $P(n)$ denote the assertion

$$\text{for all positive integers } k \text{ with } k \leq n,\ f(k) \leq 2^k.$$

We have already seen that $P(n)$ is true whenever $n \in \{1, 2, 3, 4\}$.

I now prove by induction that for all integers n with $n \geq 4$ that $P(n)$ is true.

The base case is now the case $n = 4$. $P(4)$ was proved in the exploration.

Suppose the $n \in \{k \text{ in } \mathbb{N} \ : \ k \geq 4\}$ and $P(n)$ is true. Note that since $n \geq 4$, $n - 1 \in \mathbb{N}$ and $n - 2 \in \mathbb{N}$ and $n - 3 \in \mathbb{N}$. We will deduce that $P(n+1)$ is also true. By $P(n)$

$$f(n) \leq 2^n \qquad f(n-1) \leq 2^{n-1} \text{ and } f(n-2) \leq 2^{n-2}.$$

Thus

$$\begin{aligned} f(n+1) &= f(n) + f(n-1) + f(n-2) \\ &\leq 2^n + 2^{n-1} + 2^{n-2} = 2^{n-2}\left(2^2 + 2^1 + 1\right) \\ &\leq 2^{n-2}(7) < 2^{n-2} \cdot 8 = 2^{(n-2)+3} = 2^{n+1} \end{aligned}$$

By the induction hypothesis $P(n)$ we know that $f(k) \leq 2^k$ whenever $k \leq n$. We have just shown that $f(k) \leq 2^k$ whenever $k = n + 1$. Thus $P(n + 1)$ follows.

0.4 #32 TASK: Suppose that $n \in \mathbb{N}$. Let P_n denote the set of all polynomials of degree exactly n and integer coefficients. Show that P_n is countable.
EXPLORATION We will try to use the results of Section 0.4 to avoid doing hard work. So we know that

(Cor 0.15) any subset of a countable set is countable;

(Thm 0.16) the Cartesian product of two countable sets is countable, and thus by a simple induction the cartesian product of any finite number of countable sets is countable;

(Thm 0.17) a countable union of countable sets is countable.

PROOF A polynomial of degree n with integer coefficients is a function of the form

$$g(x) = \sum_{k=0}^{n} c_k \; x^k$$

where each $c_k \in \mathbb{Z}$ and $c_n \neq 0$. Thus there is a one-one function f from P_n onto $\mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z} \times (\mathbb{Z} - \{0\})$ where we have n copies of $\mathbb{Z}$. This f is given by

$$f\left(\sum_{k=0}^{n} c_k \; x^k\right) = (c_0, c_1, ..., c_n)$$

Two polynomials are equal if and only if their ordered strings of coefficients are equal. So this function f is indeed one to one. By definition of degree n the function f is onto. Now the Cartesian product of n copies of $\mathbb{Z}$ and one copy of $\mathbb{Z} - \{0\}$ is a product of a finite number of countable sets, so $P_n \tilde{}$ a countable set and is thus countable.

0.4 #38 TASK Suppose that $a < b$ and $c < d$. Show that $[a, b]\tilde{}[c, d]$.

REMARK The statement is not true in the generality used in the text. The interval $[0, 0]$ is certainly not equivalent to the interval $[0, 1]$ – the first contains one and only one element, namely 0; the second is infinite since it contains the subset $\{1/k \ : \ k \in \mathbb{N}\}$ which is not finite.

PROOF It is easy to construct a polynomial function of degree 1 that maps $[a, b]$ one to one onto $[c, d]$. The graph of this polynomial is the straight line segment with endpoints (a, c) and (b, d). Take

$$m = \frac{d - c}{b - a} \qquad \text{and} \qquad f(x) = b + m(x - a)$$

Since $m > 0$ it is easy to see that

$$\text{whenever } a \le r < s \le b \text{ then } c = f(a) \le f(r) < f(s) \le f(b) = d$$

and thus that f maps $[a, b]$ one-to-one into $[c, d]$. It remains to show that f is onto. Consider an arbitrary y in $[c, d]$. I need to show that there is an x in $[a, b]$ such that $f(x) = y$. Now for any real x

$$f(x) = y \Longleftrightarrow m(x - a) + c = y \Longleftrightarrow \frac{y - c}{m} = x - a \Longleftrightarrow x = a + \frac{y - c}{m}$$

We are done as soon as we see why $a + (y - c)/m \in [a, b]$. Since $y \le [c, d]$ and $m > 0$ we get

$$c \ \le \ y \le d \text{ and so } 0 \le \frac{y - c}{m} \le \frac{d - c}{m} = b - a$$

$$\text{and so } a \ \le \ a + \frac{y - c}{m} \le b \text{ which means } a + \frac{y - c}{m} \in [a, b].$$

0.5 #41 TASK Suppose that $0 < a < b$. Show that $0 < a^2 < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

REMARK For this problem we will assume that every positive real r have a unique positive real square root denoted by $\sqrt{r}$.

PROOF

Step 1. Show that $0 < a^2$. This follows by the order axiom that says the product of positive reals is positive.

Step 2. Show that $a^2 < b^2$. By hypothesis, $b - a$ is positive. Now

$$b^2 = [a + (b - a)]^2 = a^2 + 2 \cdot a \cdot (b - a) + (b - a)^2$$

Note that both $2 \cdot a \cdot (b - a)$ and $(b - a)^2$ are positive since they are products of positive reals. Thus

$$2 \cdot a \cdot (b - a) + (b - a)^2 > 0$$

and

$$b^2 = a^2 + 2 \cdot a \cdot (b - a) + (b - a)^2 > a^2.$$

Step 3 Show that $0 < \sqrt{a} < \sqrt{b}$. By the meaning of $\sqrt{a}$ we know $\sqrt{a} > 0$. To get the second inequality we appeal to trichotomy.

Suppose $\sqrt{a} = \sqrt{b}$. Then

$$a = \left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2 = b, \text{ which is false.}$$

So we learn that $\sqrt{a} \neq \sqrt{b}$.

Suppose that $\sqrt{b} < \sqrt{a}$. Then by the argument of Step 2 we would learn that

$$b = \left(\sqrt{b}\right)^2 < \left(\sqrt{a}\right)^2 = a, \text{ which is false.}$$

So we learn that $\sqrt{b} \not< \sqrt{a}$.

We must conclude then that $\sqrt{a} < \sqrt{b}$.

0.5 #44 TASK Suppose that $x = lub(S)$. Show that for each positive ε there is an element s in S such that $x - \varepsilon < s \leq x$.

REMARK Implicit in the hypothesis are the assumptions that $\phi \neq S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

PROOF Consider and arbitrary positive ε. Since $x = \min(\mathcal{UB}(S))$ and $x - \varepsilon < x$ we know that $x - \varepsilon$ is not an upper bound for S. Thus there must be an s with the two properties $s \in S$ and $x - \varepsilon < s$. Pick one such and call it s_o. Since $s_o \in S$, we also know that s_o has the property that $s_o \leq x$. Thus there is an element in S, namely s_o, such that $x - \varepsilon < s_o \leq x$.