Group Representations

Let G be a group. We say that G *acts on* a set X (on the left) if there is a set map $G \times X \to X$, sending (g, x) to $g \cdot x \in X$, such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$.

Now fix a field F. A vector space V over F is called a *G-module* (or *representation* of G) if the group G acts on the set V, and if for each $g \in G$ there is a linear transformation $\sigma(g) : V \to V$ such that $g \cdot x = \sigma(g)(x)$ for all $x \in V$. A *trivial representation* is a representation with $g \cdot x = x$ for all $g \in G$.

Let $GL(V)$ denote the group of linear automorphisms of V; if $V = F^d$ then $GL(V) = GL_d(F)$. If V is a G-module then $\sigma : G \to GL(V)$ is a group homomorphism. Conversely, any group homomorphism $\sigma : G \longrightarrow GL(V)$ makes V into a G-module. Some authors take this as the definition of representation.

A representation V of G is also the same thing as a module over the ring FG. Here the *group ring* of G is a vector space FG with basis G, made into a ring with the product $(\sum \alpha_i g_i)(\sum \beta_j h_j) = \sum (\alpha_i \beta_j)(g_i h_j)$, $\alpha_i, \beta_j \in F$ and $g_i, h_j \in G$.

A *G-map* (= homomorphism of G-modules) is a linear transformation $f : V \to W$ commuting with the action of G in the sense that $f(g \cdot v) = g \cdot f(v)$. Of course, this is the same thing as a homomorphism of modules over the ring FG.

Permutation representations. Let the set X be a basis of a vector space V. Any action of G on X can be extended linearly into an action of G on V; such a representation is called a *permutation representation* because G permutes the basis. Each matrix $\sigma(g)$ consists of 0's and 1's. The *regular representation* is an example: V is the group ring FG, $X = G$ and if $v = \sum a_i g_i$ then $g \cdot v = \sum a_i (g g_i)$.

1-dimensional representations. A 1-dimensional representation (of G on F) is equivalent to a group map $G \xrightarrow{\sigma} F^*$. Since F^* is an abelian group, the commutator subgroup $[G, G]$ must map to 1, so the representation factors through $G \longrightarrow G/[G, G]$. If G has n elements each $\sigma(g)$ must be an n^{th} root of unity, because $g^n = 1$ in G. The absence of n^{th} roots of unity in F can affect the existence of 1-dimensional representations.

Let C_n denote the cyclic group of order n, with generator θ. It follows that the 1-dimensional representations of C_n (over F) are in 1-1 correspondence with the set of n^{th} roots of unity ζ in F (take $\sigma(\theta) = \zeta$). The group C_2 has two 1-dimensional representations: the trivial representation and the *sign* representation $(\theta \cdot a = -a)$. The cyclic group C_3 has three 1-dimensional representations if $F = \mathbb{C}$, but only one if $F = \mathbb{R}$.

Operations. Standard operations on vector spaces $(\oplus, \otimes, \Lambda^*$, etc.) also induce operations on G-modules. Let $V = F^m$ and $W = F^n$ be two representations. The direct sum $V \oplus W = F^{m+n}$ is a representation with $g \cdot (v + w) = (g \cdot v) + (g \cdot w)$, and the tensor product $V \otimes W = F^{mn}$ is a representation with $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$.

Let $\Lambda^d V$ denote the d^{th} exterior product of V, i.e., the vector space of dimension $\binom{m}{d}$ consisting of all alternating d-forms $v_1 \wedge \cdots \wedge v_d$ on V. The action of G on $\Lambda^d V$ is given by the formula $g \cdot (x_1 \wedge \cdots \wedge x_d) = (g \cdot x_1) \wedge \cdots \wedge (g \cdot x_d)$. For example, if $d = m$ then under the usual identification of $\Lambda^m F^m$ with F the action of g on F is multiplication by $\det(\sigma(g))$.

Definition. A nonzero G-module V is called *irreducible* (= a *simple* module) if no proper subspace is a G-submodule. V is called *completely reducible* (= *semisimple*) if it is a direct sum of irreducible G-modules.

Clearly, every 1-dimensional representation is irreducible. If $\dim(V) = 2$, there is a simple test for irreducibility: V is irreducible if no vector $v \neq 0$ in V is an eigenvalue for all of the 2×2 matrices $\sigma(g)$, $g \in G$.

Here is a general test to see if V is irreducible. For every $v \neq 0$, does the orbit of v (the set $G \cdot v = \{g \cdot v, g \in G\}$, which includes $1 \cdot v = v$) span V? If so, V is irreducible. If not, the span of $G \cdot v$ is a proper G-submodule.

Examples. 1) The regular representation of $C_2 = \{1, \theta\}$ on the plane is given by $\theta(x, y) = (y, x)$. The two vectors $(1, \pm 1)$ are eigenvectors so this representation is the direct sum $F(1, 1) \oplus F(1, -1)$.

2) The dihedral group D_n ($n \geq 3$) is defined as the group of isometries in the plane fixing the regular n-gon; the 2-dimensional representation defining D_n is irreducible (as the reflections have different eigenspaces, or because each v and its rotate by $2\pi/n$ span the plane).

3) The quaternionic group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ has an obvious 4-dimensional representation on the quaternions $\mathbb{H}$. (We take $F = \mathbb{R}$.) If $v \neq 0$, I claim that $\{v, iv, jv, kv\}$ is a basis of $\mathbb{H}$; this shows that $\mathbb{H}$ is an irreducible representation of Q. To show this, suppose given $a_i \in \mathbb{R}$ such that $a_1 v + a_2(iv) + a_3(jv) + a_4(kv) = 0$. Multiplying on the right by v^{-1} yields $a_1 + a_2 i + a_3 j + a_4 k = 0$, so all the $a_i = 0$.

3) The symmetric group S_4 acts on the regular tetrahedron in $\mathbb{R}^3$ by permuting the 4 vertices. This extends by linearity to an action of S_4 on $\mathbb{R}^3$, which is irreducible (exercise!). More generally, S_n acts on the regular n-simplex in $\mathbb{R}^{n-1}$, giving an irreducible $(n-1)$-dimensional representation of S_n.

Schur's Lemma. *1) V is irreducible $\Leftrightarrow$ $V \cong (FG)/I$ for some maximal left ideal.*
2) If V, W are irreducible, any nonzero G-map $f : V \to W$ is an isomorphism.
3) If V is irreducible, the ring $\Delta = End_G(V)$ of all G-maps $V \to V$ is a division algebra. (A division algebra is an F-algebra in which every non-zero element is a unit.) If F is algebraically closed then $\Delta = F$ (multiplication by scalars).

Proof. 1) Any choice of $v \neq 0$ in V yields a nonzero G-map $FG \to V$ sending 1 to v. Its kernel I is a left ideal, so its image is FG/I. This is a nonzero G-submodule of V, and every ideal J containing I yields a submodule J/I of V. If V is irreducible we must have $FG/I \cong V$ with no ideals J containing I. 2) If W is irreducible and $f \neq 0$, $f(V)$ must be W and $\ker(f) \neq V$. When V is irreducible this forces $\ker(f) = 0$, which means that f is an isomorphism. 3) Any nonzero G-map $f : V \to V$ must be an isomorphism by 2), in which case f^{-1} exists and is a G-map. This proves that every nonzero element f is invertible in Δ.

Examples with $F = \mathbb{R}$. The only (finite-dimensional) division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. (Over $F = \mathbb{C}$ the only f.d. division algebra is $\mathbb{C}$ itself.)

1) Consider the rotation representation of C_3 on the plane $\mathbb{R}^2$. The 2×2 matrices commuting with this action are products of scaling by r and rotation by α:

$$\begin{pmatrix} r\,cos(\alpha) & r\,sin(\alpha) \\ -r\,sin(\alpha) & r\,cos(\alpha) \end{pmatrix} = r\,cos(\alpha) + ir\,sin(\alpha), \qquad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

These form a subring Δ of $M_2(\mathbb{R})$ isomorphic to $\mathbb{C}$.

2) For the canonical representation of the quaternion group Q on $\mathbb{H}$, we have $\Delta = \mathbb{H}$. (Of course!)

Corollary to Schur's Lemma. *If $W \subset V$ is a submodule and V is completely reducible, then $V = W \oplus W'$ for some complementary submodule W'.*

Proof. Write $V = \oplus V_\alpha$ with V_α irreducible. By Zorn's lemma, there is a largest family $\{\alpha_i\}$ so that $W \cap \oplus V_{\alpha_i} = 0$; set $W' = \oplus V_{\alpha_i}$. If $W \oplus W'$ isn't V, it doesn't contain some V_β; this would imply $(W \oplus W') \cap V_\beta = 0$, leading to a contradiction.

Remark. The regular representation is never irreducible (unless $G = 1$). To see this, recall that the norm element of FG is the sum $N = \sum g$ of every element in G. Since $gN = N$ for all $g \in G$, N generates a 1-dimensional submodule $F \cdot N$ of FG.

The next theorem states that FG is completely reducible (when $1/|G|$ exists in F), and that it contains *every* irreducible representation at least once. Therefore $FG = F \cdot N \oplus W'$ for some W'. In fact, since $N^2 = |G| \cdot N$, the element $e = N/|G|$ is an idempotent of the ring FG and $FG = W \oplus W'$ with $W' = FG(1-e)$.

The hypothesis (that $1/|G|$ exists in F) fails only when F has characteristic $p > 0$ and p divides $|G|$. In this case, the regular representation is *never* completely reducible, because $F \cdot N \subset FG$ has no complement. (If $FG = F \cdot N \oplus W'$ then some nonzero multiple of N must be idempotent, which is impossible because $N^2 = |G| \cdot N = 0$.)

Maschke's Theorem. *If G is a finite group and $\frac{1}{|G|} \in F$, then:*

1) Every representation of G is completely reducible.

2) There are only a finite number s of irreducible representations V_i (up to isomorphism), with V_i occuring $n_i \geq 1$ times in the regular representation of G on FG.

We will write Δ_i for the division algebra $End_G(V_i)$, and set $d_i = \dim_F(\Delta_i)$.

3) The i^{th} irreducible representation has dimension $d_i n_i$. Hence

$$|G| = \sum_{i=1}^{s} d_i n_i^2.$$

4) FG is the product of the s matrix rings $M_{n_i}(\Delta_i)$. The projection $FG \to M_{n_i}(\Delta_i)$ allows us to identify V_i with the $M_{n_i}(\Delta_i)$-module $\Delta_i^{n_i}$.

Some explanation of parts 3) and 4) is in order. The matrix ring $M_{n_i}(\Delta_i)$ is the direct sum of its n_i columns, each being the irreducible representation $V_i = \Delta_i^{n_i}$ of dimension $d_i n_i$ over F. Summing over $i = 1, \ldots, s$ yields the decomposition of the $|G|$-dimensional representation FG into its irreducible components.

Corollary (Complex representations). *Suppose that $F = \mathbb{C}$. Then*

$$|G| = \sum n_i^2, \text{ where } n_i = \dim(V_i).$$

If G is abelian then there are exactly $|G|$ irreducible representations, all of them 1-dimensional. Every representation is a direct sum of 1-dimensional representations.

Indeed, we must have $\Delta_i = \mathbb{C}$ and $FG = M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_s}(\mathbb{C})$. If G is abelian, FG is a commutative ring; this forces $n_1 = \cdots = n_s = 1$, or $FG = \Pi_{i=1}^{s}\mathbb{C}$.

Proposition. *Let c denote the number of conjugacy classes of elements of G. If $F = \mathbb{C}$ then there are c irreducible representations of G. In general, the center E_i of Δ_i is a finite field extension of F, and $c = \sum_{i=1}^{s} \dim_F(E_i) \geq s$.*

The connection to Maschke's theorem comes from the observation that the center of FG is $\prod E_i$. Write $C_1, \ldots, C_c$ for the conjugacy classes of G. The c elements $z_j = \sum\{g \in C_j\}$ are central elements of FG, and form a basis for the center of FG.

Examples. 1) If $F = \mathbb{C}$ then C_3 has three irreducible 1-dimensional representations. If $F = \mathbb{R}$ then C_3 has only two irreducible representations: the trivial representation $V_1 = \mathbb{R}$ and the rotation representation on the plane $V_2 = \mathbb{R}^2$.

2) The dihedral group $D_2 = C_2 \times C_2$ is abelian, so it has 4 one-dimensional representations–even over $\mathbb{R}$. The regular representation FD_2 is the sum of these 4 representations. Finding the irreducible representations of D_3 and D_5 is an exercise.

3) The dihedral group D_4 has 8 elements, and $D_4/[D_4, D_4]$ is $C_2 \times C_2$. Thus it has exactly 4 one-dimensional representations. We have already observed that D_4 has a 2-dimensional irreducible representation V as its "birth certificate". Since $8 = 4 \cdot 1 + 2^2$, this accounts for all the irreducible representations of D_4.

4) The quaternionic group Q has 8 elements and 5 conjugacy classes. Since $Q/[Q, Q] = Q/\{\pm 1\} = C_2 \times C_2$, there are exactly 4 one-dimensional representations. Counting $(8 = 4 + 4)$ shows there is exactly one other irreducible representation V_5, of dimension 2 or 4 depending on F. If $F = \mathbb{R}$, then V_5 is the 4-dimensional representation of Q on $\mathbb{H}$; if $F = \mathbb{C}$ then V_5 is the 2-dimensional representation of Q on $\mathbb{H} \cong \mathbb{C}^2$ (and $n_5 = 2$).

Exercises. 1) Consider the rotation representation of C_3 on the complex plane $\mathbb{C}^2$. Write this as the direct sum of two 1-dimensional representations over $F = \mathbb{C}$.

2) Provide details for the sketch given above that the 2-dimensional representation of D_n is irreducible when $n \geq 3$.

3) Describe all irreducible representations of D_3 and D_5 over $\mathbb{R}$ and over $\mathbb{C}$. *Hint:* Find two actions of D_5 on the regular pentagon.

4) Prove that the 3-dimensional representation of S_4 arising from the action on the regular tetrahedron is irreducible.

5) Determine all irreducible complex representations of the alternating group A_4 (12 elements). *Hint.* Use the fact that $[A_4, A_4]$ has 4 elements to write down all group maps $A_4 \to \mathbb{C}^*$. Then let G act on the set $X = \{(12)(34), (13)(24), (14)(23)\}$ of elements of A_4 by conjugation, and prove that FX is irreducible.

6) If V is irreducible and W is any 1-dimensional representation of G, show that the tensor product $V \otimes W$ is also an irreducible representation of G.

Young Tableaux. Let S_n denote the symmetric group on n elements. The number of conjugacy classes of S_n equals the number of unordered partitions of n; the unordered partition $\lambda = \{r_1, ..., r_h\}$ corresponds to the congugacy class of $(1, \ldots, r_1) \ldots (n+1-r_h, \ldots, n)$. Since the order of the r_i doesn't matter, we always assume that $r_1 \geq r_2 \geq \ldots \geq r_h$. Each partition λ determines an arrangement of n empty boxes into h rows, the i^{th} row has r_i boxes; such an arrangement is called a

Young Tableau of shape λ and size n. The corresponding irreducible representation S^λ of S_n is sometimes called the *Specht module* of λ. We shall write f^λ for $\dim(S^\lambda)$.

If we fill in the boxes of a Young tableau of shape λ with the numbers $1, ..., n$ we get a *Young diagram* D. We call D *standard* if a) the entries in every row are increasing, and b) the entries in every column are increasing. The number of standard Young diagrams of shape λ equals $f^\lambda = \dim S^\lambda$.

There is a simple product formula for f^λ, called the *hook formula.* If (i,j) is a box in a Young Tableau, the corresponding *hooklength* h_{ij} is the number of boxes in the "hook" $\{(i,k), k \geq j\} \cup \{(k,j), k \geq i\}$ with vertex (i,j). The hook formula says that

$$\dim(S^\lambda) = f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}}.$$

If R (resp. C) denotes the subgroup of S_n consisting of permutations which merely permute the entries in the rows (resp. in the columns) of D, then the Specht module may be described as $S^\lambda = (FS_n)f_D \subseteq FS_n$, where $f_D \in FS_n$ is the sum

$$f_D = \sum_{\substack{\tau\in C \\ \sigma\in R}} (-1)^\tau \tau\sigma.$$

Representations of S_4. The only partitions of $n = 4$ are $\{1,1,1,1,\}$, $\{2,1,1\}$, $\{2,2\}$, $\{3,1\}$ and $\{4\}$, corresponding to the 5 Young tableau of size 4. Therefore there are exactly 5 irreducible representations of S_4. The only way to add up to 24 using five squares is $24 = 1{+}1{+}4{+}9{+}9$, so S_4 has two irreducible 3-dimensional representations (corresponding to two actions of S_4 on the regular tetrahedron), one irreducible 2-dimensional representation (S_4 acts on the triangle in the plane by $S_4 \to D_3 \subset GL_2(F)$) and two 1-dimensional representations (the trivial representation and the sign representation). Of course, the dimensions of these representations can also be found by the hook formula.

Characters of finite groups. For simplicity, we concentrate on representations of a finite group G over $\mathbb{C}$. The *character* χ_V of a representation $V = \mathbb{C}^n$ is defined to be the set map $\chi_V : G \to \mathbb{C}$ sending g to the trace of the matrix $\sigma(g)$. This map is independent of the choice of basis for V, since the trace is independent of this choice. This also shows that if V and W are isomorphic representations then $\chi_V = \chi_W$. Note that χ_V determines the dimension of V, because $\chi_V(1) = \mathrm{trace}(I) = \dim(V)$. We will see that in fact χ_V completely determines V (over $F = \mathbb{C}$).

Examples. 1) Let V be the 2-dimensional rotation representation of the cyclic group C_n. Then $\chi_V(\theta^k) = 2\cos(2\pi k/n)$ for all k.
2) The character of the regular representation $V = \mathbb{C}G$ is easy to work out. The matrix $\sigma(g)$ consists of 0's and 1's, and the (i,i) entry is 1 exactly when $g \cdot g_i = g_i$ in G. This never happens when $g \neq 1$, meaning that all diagonal entries are 0, and so the trace is 0. In conclusion, if $g \neq 1$ then $\chi_{\mathbb{C}G}(g) = 0$.
3) The character of a 1-dimensional representation V is $\chi_V(g) = \sigma(g)$, simply because the trace of a 1×1 matrix (a) is a. These characters are not so interesting.

The characters $\chi_1, ..., \chi_s$ of the irreducible representations $V_1, ..., V_s$ are called the "irreducible" characters. Every character χ_V is a linear combination of the irreducible characters in the vector space $\mathbb{C}^G$ of all set maps $G \to \mathbb{C}$. To see this, write V as the sum $V = V_{i_1} \oplus \cdots \oplus V_{i_r}$ of irreducible representations. This puts the matrices $\sigma(g)$ in block diagonal form, and we have $\chi_V(g) = \chi_{i_1}(g) + \cdots + \chi_{i_r}(g)$ for all $g \in G$. In particular, the character of the regular representation is $\sum n_i \chi_i$, from which we deduce that for every $g \neq 1$ we have $\sum n_i \chi_i(g) = 0$. (And of course $\sum n_i \chi_i(1) = \sum n_i^2 = |G|$.)

A *class function* on G is a function $\phi : G \to \mathbb{C}$ which is constant on conjugacy classes, i.e., if $g' = hgh^{-1}$ then $\phi(g') = \phi(g)$. The characters χ_V are examples of class functions, because $\chi_V(g)$ and $\chi_V(g')$ are the traces of the matrices $\sigma(g)$ and $\sigma(g') = P\sigma(g)P^{-1}$, $P = \sigma(h)$. We choose representatives $g_1, \ldots, g_c$ of the c conjugacy classes of G; any class function (including χ_V) is then completely determined by its values on these c elements. Thus the class functions form a c-dimensional vector subspace of $\mathbb{C}^G$. We are going to show that the irreducible characters χ_i of G form a basis of this vector space. Since they belong to this subspace, and there are $s = c$ of them, it suffices to show that they are linearly independent. This follows from the following result, whose proof we omit.

Orthogonality Relations. *The irreducible characters are orthogonal (with respect to the usual hermitian inner product) in the vector space* $\mathbb{C}^G$*:*

$$\langle \chi_i | \chi_j \rangle = \sum_{g \in G} \chi_i(g)^* \chi_j(g) = \begin{cases} |G| & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Corollary. *The irreducible characters* $\{\chi_1, \ldots, \chi_c\}$ *form a basis for the vector space of class functions on* G*.*

Thus every class function ϕ *uniquely determines complex numbers* $a_1, \ldots, a_c$ *such that* $\phi(g) = \sum a_i \chi_i(g)$ *for all* $g \in G$*. In fact,* $a_i = \frac{1}{|G|}\langle \phi | \chi_i \rangle$*. In particular, the character* χ_V *of any representation* V *uniquely determines integers* m_i *such that*

$$V \cong \overbrace{V_1 \oplus \ldots V_1}^{m_1} \oplus \overbrace{V_2 \oplus \ldots V_2}^{m_2} \oplus \cdots \oplus \overbrace{V_c \oplus \ldots V_c}^{m_c}.$$

Character tables. The complex numbers $\chi_i(g_j)$ assemble to form a $c \times c$ matrix, called the *character table* of G. The above results state that the character table tells us almost everything about all representations of G. The Orthogonality Relations imply that the columns are linearly independent (being orthogonal). The character table is not quite a unitary matrix; it satisfies the relation $A^{*t}A = |G| \cdot I$ instead.

• The character tables of C_2 and C_3 are $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$, $\omega = e^{2\pi i/3}$.

• For S_3 there are 3 conjugacy classes, represented by: $\{1, (12), (123)\}$. The character table for S_3 is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$