Algebra I, Fall 2015, Homework 3

Due: Thursday, October 22, in class.

Problem 1:

(a) Let G be a finite group and let X be a finite left G-set. For each $g \in G$ let ${}^gX = \{x \in X \mid g.x = x\}$ be the set of points in X that are fixed by g. Prove the Cauchy-Frobenius formula for the number of orbits:

$$|G\backslash X| \;=\; \frac{1}{|G|} \sum_{g \in G} |{}^gX|\,.$$

Hint: Consider the set $\{(g,x) \in G \times X \mid g.x = x\}$.

(b) Prove that there are $\frac{1}{12}(m^4 + 11m^2)$ ways to color the sides of a regular tetrahedron with m colors, up to rotation.

(c) Find the number of ways to color the sides of a cube with m colors, up to rotation.

Problem 2:

(a) Prove that there are exactly two isomorphism classes of groups of order 6.

(b) Prove that there are exactly five isomorphism classes of groups of order 12.

Hint: Any group of order 12 contains a normal Sylow subgroup.

Problems from Basic Algebra 1:

2.3: 2

2.5: 8

4.6: 8, 9, 11

Hints: For 9: Use 8 and 1.13(1). For 11: If P and Q are two normal subgroups of G such that $P \cap Q = \{1\}$, then $PQ \cong P \times Q$. If P is any p-group then use the homomorphism $P \to P/C(P)$ and induction to prove that P is nilpotent.