Algebra I, Fall 2015, Homework 5

Due: Thursday, December 3, in class.

Problem 1: (Cayley-Hamilton)

Let R be a commutative ring, let M an R-module generated by n elements, and let $\phi : M \to M$ be an R-homomorphism. Then there exists a monic polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ in $R[x]$ such that $p(\phi) = 0$ holds in $\mathrm{End}_R(M)$.

Hint: Let M be generated by $\{m_1, m_2, \ldots, m_n\}$, write $\phi(m_j) = \sum_i a_{ij} m_i$, and set $A = (a_{ij}) \in M_n(R)$. Then show that $p(\phi) = 0$ where $p(x) = \det(xI_n - A)$. You can use the equation in M^n, where the matrix has coefficients in $R[\phi]$:

$$(\phi I_n - A)\begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Problem 2:

Let R be a commutative ring and let $\phi : R^m \to R^n$ be an injective homomorphism of R-modules. Show that $m \leq n$.

Hint: Assume that $\phi : R^n \to R^n$ is an injective R-homomorphism such that $\phi(R^n) \subset \mathrm{Span}_R\{e_1, \ldots, e_{n-1}\}$. Show that ϕ satisfies an equation of the form $\phi^k + a_{k-1}\phi^{k-1} + \cdots + a_1\phi + a_0$ in $\mathrm{End}_R(R^n)$, where $a_i \in R$ and $a_0 \neq 0$. Now apply this equation to e_n.

Problems from Basic Algebra 1:

3.6: 2
3.7: 2
3.8: 1
3.10: 2, 5, 6