Algebra 551 Final Exam, Fall 2015

Instructions: Do 5 of the following problems. Give complete proofs of your claims. You may quote theorems from the course without proof. Your solutions will be evaluated for correctness, completeness, and clarity, so please write your solutions carefully, clearly, and legibly.

Problem 1.
Let G be a group of order 10000. Prove that G is not simple. (Hint: Prove first that G is not isomorphic to a subgroup of S_{16}.)

Problem 2.
Prove that the function $\phi(a+ib) = a^2 + b^2$ makes the ring $\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}]$ of Gaussian integers into an Euclidean domain.

Problem 3.
Let A be a real symmetric matrix. Prove that all eigenvalues of A are non-negative if and only if there exists a real matrix P such that $A = P^T P$.

Problem 4.
Find the cardinality of any maximal set of non-similar nilpotent matrices in $M_7(\mathbb{C})$.

Problem 5.
Let $\phi : \mathbb{Z}^3 \to \mathbb{Z}^4$ be the homomorphism of abelian groups defined by

$$\begin{aligned} \phi(e_1) &= 6e_1 + 3e_2 + 3e_3 + 3e_4 \\ \phi(e_2) &= 4e_1 + 3e_2 + e_3 + e_4 \\ \phi(e_3) &= 2e_1 + 3e_2 + 17e_3 + 8e_4 \end{aligned}$$

Find a product of cyclic groups that is isomorphic to the cokernel $\mathbb{Z}^4/\phi(\mathbb{Z}^3)$.

Problem 6.
Let $A = \mathbb{C}[S_3]$ be the group ring of the symmetric group S_3 over the field of complex numbers. Find a direct product of matrix rings over division rings that is isomorphic to A.

Problem 7.
Let p be a prime number. Classify, up to isomorphism, all groups of order $2p$.