Alg. Geo. Class

R commutative ring.

Noetherian : Every ideal $I \subseteq R$ fin. gen

$\Updownarrow$ Ascending chain cond. on ideals

$\Updownarrow$ Every non-empty family of ideals has max elt.

Hilbert basis Thm

R Noetherian $\Rightarrow$ $R[x]$ Noetherian.

Pf Assume $I \subseteq R[x]$ NOT f.g.

Choose $f_1 \in I$, $f_1 \neq 0$, of minimal degree.

Given $f_1, \ldots, f_{i-1}$, choose $f_i \in I \setminus \langle f_1, \ldots, f_{i-1} \rangle$ of min deg.

a_i = lead coef of f_i.

$J = \langle a_1, a_2, a_3, \ldots \rangle \subseteq R$

R Noetherian $\Rightarrow$ $J = \langle a_1, \ldots, a_m \rangle$

Write $a_m = \sum_{i=1}^{m} r_i a_i$, $r_i \in R$.

$$f' = f_{m+1} - \sum_{i=1}^{m} r_i f_i \cdot x^{\deg(f_{m+1}) - \deg(f_i)}$$

Then $f' \in I \setminus \langle f_1, \ldots, f_m \rangle$ and $\deg(f') < \deg(f_{m+1})$ ↯.

□

Cor k field $\Rightarrow$ $k[x_1, \ldots, x_n]$ Noetherian.

$k[x_1, \ldots, x_n]/I$ Noeth for any ideal I.

Def An affine ring over k is a f.g. commutative k-algebra.

$R \cong k[x_1, \ldots, x_n]/I$

<u>R-module M</u>: Additive group $(M,+)$ with mult. map

$R\times M \longrightarrow M$, $(r,m)\longmapsto rm$ $\quad r_1(r_2 m) = (r_1 r_2)m$

$1.m = m$, $(r_1+r_2)m = r_1 m + r_2 m$, $r(m_1+m_2) = rm_1 + rm_2$.

M <u>finitely generated</u>: $\exists$ finite subset $\{m_1, m_2, \dots, m_s\} \subseteq M$ s.t.

$$M = \{r_1 m_1 + \dots + r_s m_s \mid r_1, \dots, r_s \in R\}.$$

<u>Exercise</u> $R \subseteq S \subseteq T$ subrings.

S f.g. R-module and T f.g. S-module $\Rightarrow$ T f.g. R-module

<u>Exercise</u> $R \subseteq S$ subring. S f.g. R-module $\Rightarrow$ S integral over R.

Every $s \in S$ satisfies poly eqn. $s^p + r_1 s^{p-1} + \dots + r_p = 0$, $r_i \in R$.

(see page 8.)

<u>Note</u> $f \in k[x_1, \dots, x_n]$.

$f = f_0 + f_1 x_n + \dots + f_d x_n^d$, $\quad f_i \in k[x_1, \dots, x_{n-1}]$, $f_d \neq 0$.

f monic in x_n (or $f_d \in k$) $\Rightarrow$

$k[x_1, \dots, x_n]$ f.g. module over subring $k[x_1, \dots, x_{n-1}, f]$

generated by $\{1, x_n, \dots, x_n^{d-1}\}$.

<u>Noether's Normalization Thm</u>

Every affine ring is a f.g. module over a polynomial subring.

I.e. R affine over k $\Rightarrow$ $\exists$ subring $S \subseteq R$: R f.g. S-module

and $S \cong k[x_1, \dots, x_r]$.

<u>Proof</u> Induction over # generators of R as k-algebra.

Zero: $R = k$, ok!

n: $R = k[x_1, \dots, x_n]/I$. $\quad$ WLOG $I \neq 0$.

Choose $0 \neq f \in I$.

<u>Easy case</u>: If f monic in x_n, then

$k[x_1, \ldots, x_n]$ f.g. module over $T = k[x_1, \ldots, x_{n-1}, f]$

$\Rightarrow$ $R = k[x_1, \ldots, x_n]/I$ f.g. module over $T/I \cap T \subseteq R$.

$T/I \cap T$ k-algebra gen. by $\bar{x}_1, \ldots, \bar{x}_{n-1}$, $\bar{f} = 0$.

Induction $\Rightarrow$ $T/I \cap T$ f.g. module over poly subring.

<u>General case</u>:

$f = \sum c_{\underline{a}} x^{\underline{a}}$, $\quad x^{\underline{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, $c_{\underline{a}} \in k$.

Choose $e \in \mathbb{N}$ such that

$\forall \underline{a} \in \mathbb{Z}^n : c_{\underline{a}} \neq 0 \Rightarrow e > \max\{a_1, a_2, \ldots, a_n\}$.

Set $x_i' = x_i - x_n^{e^i}$ for $1 \le i \le n-1$

$k[x_1, \ldots, x_n] = k[x_1', \ldots, x_{n-1}', x_n]$.

<u>Claim</u>: f is monic in x_n as poly in $k[x_1', \ldots, x_{n-1}', x_n]$.

$x_1^{a_1} \cdots x_n^{a_n} = (x_1' + x_n^{e})^{a_1} (x_2' + x_n^{e^2})^{a_2} \cdots (x_{n-1}' + x_n^{e^{n-1}})^{a_{n-1}} x_n^{a_n}$

is monic in x_n.

leading term: $x_n^{a_n + a_1 e + \cdots + a_{n-1} e^{n-1}}$

Choice of $e \Rightarrow$ all monomials occuring in f have distinct leading terms.

$\therefore$ $f \in k[x_1', \ldots, x_{n-1}', x_n]$ monic in x_n.

$\Rightarrow$ $k[x_1, \ldots, x_n]$ f.g. module over $k[x_1', \ldots, x_{n-1}', f] = T$

$\Rightarrow$ R f.g. module over $T/I \cap T \subseteq R$

$T/I \cap T$ gen by $\overline{x_1'}, \ldots, \overline{x_{n-1}'}$ as k-alg. so f.g. module over poly subring.

k field.

k alg. closed ($k=\bar{k}$) if every non-const $f(x)\in k[x]$ has root.

Then $f(x) = c(x-a_1)(x-a_2)\cdots(x-a_d)$, $c, a_1, \dots, a_d \in k$.

Note $k=\bar{k}$ and $k\subseteq \mathbb{F}$ alg. extension $\Rightarrow$ $k=\mathbb{F}$.

Exercise $\mathbb{C}=\bar{\mathbb{C}}$.

Hints: $f(x)\in\mathbb{C}[x]$ not const.

$|f(x)|\longrightarrow\infty$ as $|x|\longrightarrow\infty$.

Continuity: $|f(x)|$ attains local min. at $x_0\in\mathbb{C}$.

Assume $f(x_0)\neq 0$.

Replace $f(x)$ with $\dfrac{f(x+x_0)}{f(x_0)}$: $|f(x)|$ has local min at $x=0$, $f(0)=1$.

$f(x) = 1 - a x^r g(x)$, $g(x)\in\mathbb{C}[x]$, $g(0)=1$.

Replace $f(x)$ with $f\left(\frac{x}{\sqrt[r]{a}}\right)$:

$f(x) = 1 - x^r h(x)$, $h(x)\in\mathbb{C}[x]$, $h(0)=1$.

Now: $x\in\mathbb{R}_+$ small $\Rightarrow$ $|f(x)| < 1 = |f(0)|$ $\lightning$.

Example p prime, $n\in\mathbb{N}$ $\Rightarrow$ $\exists$ field $\mathbb{F}_{p^n}$: $|\mathbb{F}_{p^n}| = p^n$.

$\mathbb{F}_{p^n}$ unique up to iso.

$\mathbb{F}_p\subseteq\mathbb{F}_{p^2}\subseteq\mathbb{F}_{p^3}\subseteq\cdots$

$\overline{\mathbb{F}_p} = \bigcup_{n=1}^{\infty}\mathbb{F}_{p^n}$ alg. closed field.

Example $\overline{\mathbb{Q}} = \{a\in\mathbb{C} \mid a \text{ alg. over } \mathbb{Q}\}$ alg. closed field. (countable)

Alg. Geo.

k Field. $\mathbb{A}^n = k^n = \{(a_1, \dots, a_n) \mid a_i \in k\}$ affine n-space.

Let $f = f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$

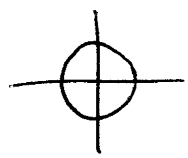
Defines function: $f : \mathbb{A}^n \longrightarrow k$, $(a_1, \dots, a_n) \longmapsto f(a_1, \dots, a_n)$.

Exercise Assume $|k| = \infty$. If $f, g \in k[x_1, \dots, x_n]$ define same function on $\mathbb{A}^n$, then $f = g$ as polynomials.

Exercise $k = \bar{k} \Rightarrow |k| = \infty$

Def $S \subseteq k[x_1, \dots, x_n]$ any subset.

$V(S) = \{a \in \mathbb{A}^n \mid f(a) = 0 \ \ \forall f \in S\}$ alg. subset of $\mathbb{A}^n$.

Examples 1) $V(x^2 + y^2 - 1)$  2) $\{(a_1, \dots, a_n)\} = V(x_1 - a_1, \dots, x_n - a_n)$

Exercises

(1) $I = \langle S \rangle \subseteq k[x_1, \dots, x_n] \Rightarrow V(S) = V(I)$.

(2) $I \subseteq J \Rightarrow V(J) \subseteq V(I)$

(3) $V\left(\bigcup_\alpha I_\alpha\right) = V\left(\sum_\alpha I_\alpha\right) = \bigcap_\alpha V(I_\alpha)$

(4) $V(I \cap J) = V(I \cdot J) = V(I) \cup V(J)$.

Zariski Topology

Zariski-closed subsets of $\mathbb{A}^n$ are the algebraic subsets.

I.e. $U \subseteq \mathbb{A}^n$ open $\Leftrightarrow \exists S : \mathbb{A}^n \setminus U = V(S)$.

(3) + (4) $\Rightarrow$ this is a topology. $\mathbb{A}^n = V(0)$, $\emptyset = V(1)$.

Example Zariski closed subsets of $\mathbb{A}^1$ are finite sets & $\mathbb{A}^1$.

Def $W \subseteq \mathbb{A}^n$ any subset.

$I(W) = \{f \in k[x_1, \dots, x_n] \mid f(a) = 0 \ \ \forall a \in W\}$ ideal of W.

Exercises (1) $V \subseteq W \Rightarrow I(W) \subseteq I(V)$

(2) $I(\emptyset) = (1) = k[x_1, \to x_n]$

(3) $I(\mathbb{A}^n) = (0)$ if $|k| = \infty$.

Note $W \subseteq \mathbb{A}^n$ subset.
$A(W) = k[x_1, \to x_n]/I(W)$
$=$ ring of poly. functions $W \to k$.

Def $I \subseteq R$ ideal.
Radical of I: $\sqrt{I} = \{f \in R \mid f^m \in I$ for some $m\}$

I is a radical ideal if $\sqrt{I} = I$.

Exercise $\sqrt{I}$ is a radical ideal.

Claim $I(W)$ is radical.
Since $I(W) \subseteq \sqrt{I(W)}$, must show $\sqrt{I(W)} \subseteq I(W)$.
$f \in \sqrt{I(W)}$. Choose m s.t. $f^m \in I(W)$.
For $a \in W$ we have $f(a)^m = 0 \Rightarrow f(a) = 0$.
$\therefore f \in I(W)$.

Exercises

(1) $S \subseteq I(V(S))$, $S \subseteq k[x_1, \to x_n]$

(2) $W \subseteq V(I(W))$, $W \subseteq \mathbb{A}^n$.

(3) $W \subseteq \mathbb{A}^n$ alg. subset $\Rightarrow W = V(I(W))$

(4) $I \subseteq k[x_1, \dots, x_n]$ ideal $\Rightarrow V(I) = V(\sqrt{I})$ and $\sqrt{I} \subseteq I(V(I))$

Example $k = \mathbb{R}$, $\mathbb{A}^1 = \mathbb{R}$, $I = \langle x^2+1 \rangle \subseteq k[x]$.
$V(I) = \emptyset$. $I = \sqrt{I} \subsetneq I(V(I)) = k[x]$.

Nullstellensatz
$k = \bar{k}$, $I \subseteq k[x_1, \to x_n]$ ideal.
Then $I(V(I)) = \sqrt{I}$.

Weak Nullstellensatz

$k = \bar{k}$, $I \subsetneq k[x_1, \ldots, x_n]$ proper ideal $\Rightarrow V(I) \neq \emptyset \subseteq \mathbb{A}^n$.

Proof

WLOG I maximal, $R = k[x_1, \ldots, x_n]/I$ field.

Noether Normalization Thm $\Rightarrow$

$\exists\, y_1, \ldots, y_m \in R$ alg. indep. over k such that R f.g. module over subring $k[y_1, \ldots, y_m]$.

Claim: $m = 0$

otherwise $y_1^{-1} \in R$ is integral over $k[y_1, \ldots, y_m]$.

$y_1^{-p} + y_1^{1-p} \cdot f_1 + \cdots + f_p = 0$, $f_i \in k[y_1, \ldots, y_m]$.

$\Rightarrow 1 = -y_1 f_1 - \cdots - y_1^p f_p \in \langle y_1 \rangle \subsetneq k[y_1, \ldots, y_m]$ ↯.

$\therefore$ The field R is algebraic over k.

$k = \bar{k} \Rightarrow R = k$.

$k \subseteq k[x_1, \ldots, x_n] \longrightarrow R = k$.

a_i = image of x_i in $R = k$.

Then $x_i - a_i \in I = \text{kernel}(k[x_1, \ldots, x_n] \longrightarrow R)$

$\langle x_1 - a_1, \ldots, x_n - a_n \rangle \subseteq I \subsetneq k[x_1, \ldots, x_n]$

$\therefore I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, $V(I) = \{(a_1, \ldots, a_n)\}$.

$\square$

Thm $I \subseteq k[x_1, \ldots, x_n]$ any ideal and $k = \bar{k}$

$\Rightarrow I(V(I)) = \sqrt{I}$

Proof

Exercise: $\sqrt{I} \subseteq I(V(I))$

Let $f \in I(V(I))$. Must show $f \in \sqrt{I}$.

$\mathbb{A}^{n+1}$ has coordinates $x_1, \ldots, x_n, y$.

Set $J = \langle I, 1-yf \rangle \subseteq k[x_1, \ldots, x_n, y]$

<u>Claim:</u> $V(J) = \emptyset \subseteq \mathbb{A}^{n+1}$.

$$p = (a_1, \ldots, a_n, b) \in V(J) \Rightarrow (a_1, \ldots, a_n) \in V(I) \Rightarrow f(a_1, \ldots, a_n) = 0.$$

$$\Rightarrow (1-yf)(p) = 1 - b \cdot f(a_1, \ldots, a_n) = 1 \neq 0 \ \lightning.$$

Weak Nullstellensatz $\Rightarrow$ $J = k[x_1, \ldots, x_n, y]$

$$\mathbf{1} \in J \Rightarrow 1 = h_1 g_1 + \cdots + h_m g_m + q(1-yf)$$

where $g_i \in I$, $h_i, q \in k[x_1, \ldots, x_n, y]$.

Replace y with f^{-1}, then multiply by f^N to clear denominators:

$$f^N = \tilde{h}_1 g_1 + \cdots + \tilde{h}_m g_m$$

where $\tilde{h}_i = f^N h_i(x_1, \ldots, x_n, f^{-1}) \in k[x_1, \ldots, x_n]$

$\therefore f^N \in I$, so $f \in \sqrt{I}$.

$\square$

<u>Integral elements</u>

M R-module.

$\mathrm{Ann}(M) = \{a \in R \mid am = 0 \ \forall m \in M\}$ ideal in R.

M is <u>faithful</u> if $\mathrm{Ann}(M) = 0$.

<u>Prop</u> S commutative ring, $R \subseteq S$ subring, $\alpha \in S$. TFAE:

(1) α is integral over R.

(2) $R[\alpha]$ is a finitely generated R-module.

(3) $\exists$ faithful $R[\alpha]$-module that is finitely generated as R-module.

<u>Note</u>: S is a faithful $R[\alpha]$-module, so S f.g. R-module $\Rightarrow \alpha$ integral.

Proof

$(1) \Rightarrow (2)$: Assume $f(\alpha) = 0$ in S where $f(x) \in R[x]$ is monic of degree n. Then $R[\alpha]$ is generated by $1, \alpha, \dots, \alpha^{n-1}$ as R-module:

Any element in $R[\alpha]$ can be written as $g(\alpha)$, $g \in R[x]$.

Write $g(x) = q(x) f(x) + r(x)$, $q(x), r(x) \in R[x]$, $\deg(r) < n$.

Then $g(\alpha) = r(\alpha)$.

$(2) \Rightarrow (3)$: $R[\alpha]$ is a faithful $R[\alpha]$-module.

$(3) \Rightarrow (1)$: M faithful $R[\alpha]$-module gen. by $m_1, \dots, m_n$ as R-module.

Write $\alpha m_j = \sum_{i=1}^{n} a_{ij} m_i$, $a_{ij} \in R$.

$A = (a_{ij})$ $n \times n$ matrix.

$$(\alpha I_n - A) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{in } M^n.$$

$$\Rightarrow \det(\alpha I_n - A)\, m_i = 0 \quad \forall i$$

$$\Rightarrow \det(\alpha I_n - A) \in \mathrm{Ann}(M) = 0.$$

$\therefore$ α satisfies polynomial eqn. $\det(\alpha I_n - A) = 0$ in S, coefficients in R.

□