Algebra 2, Homework 1

Due: March 8, 2012.

Problem 1:
Let $F \subset E$ be an algebraic field extension and R a ring such that $F \subset R \subset E$. Prove that R is field.

Problem 2:
Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then E has the $\mathbb{Q}$-basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$. Find $a, b, c, d \in \mathbb{Q}$ such that $(1 + \sqrt{2} + \sqrt{3})^{-1} = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$.

Problem 3:
Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.
(a) Show that $E/\mathbb{Q}$ is Galois.
(b) Find $\mathrm{Gal}(E/\mathbb{Q})$.
(c) Find $\alpha \in E$ such that $E = \mathbb{Q}(\alpha)$.

Problem 4:
Let E be a finite extension of $\mathbb{Q}$. Show that E contains only finitely many roots of 1.

Problem 5:
Let K/F be a finite Galois extension such that $[K : F] = p^n$ where p is a prime and $n \geq 1$. Show that:
(a) There exists a subextension $F \subset E \subset K$ such that $[E : F] = p$.
(b) Any such subextension E is Galois over F.

Problem 6:
Let $F \subset E \subset K$ be field extensions such that K/F is Galois. Set $G = \mathrm{Gal}(K/F)$ and $H = \mathrm{Gal}(K/E)$. Show that $\mathrm{Aut}_F(E) \cong N_G(H)/H$.

Problem 7:
Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4 with exactly two real roots. Show that $\mathrm{Gal}(f(x)/\mathbb{Q})$ is either S_4 or D_4.

Problem 8:
Let F be a perfect field and $F \subset E$ an algebraic field extension, such that every non-constant polynomial $f(x) \in F[x]$ has a root in E. Show that E is algebraically closed. (Hint: Primitive element theorem.)