Algebra 2, Homework 2

Due: April 19, 2012.

Problem 1:
Let k be an algebraically closed field and consider the set $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$. Show that W is Zariski closed and find $I(W) \subset k[x, y, z]$.

Problem 2:
Set $I = (y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3) \subset k[x, y]$, where k is an algebraically closed field. Find the radical $\sqrt{I} \subset k[x, y]$. (Does it depend on the characteristic of k?)

Problem 3:
(a) Let K be a field and $R, S \subset K$ two normal subrings. Show that $R \cap S$ is also a normal ring.
(b) Show that if k is any field, then $R = k[x, y, z]/(z^2 - xy)$ is a normal ring. Hint: $R \cong k[x, xt, xt^2] \subset k(x, t)$.

Problem 4:
Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. Prove that $F(0) = 0$ and $F(A \oplus B) = F(A) \oplus F(B)$ for all objects $A, B \in \text{ob}(\mathcal{C})$.

Problem 5:
Let R be a ring and let $\mathcal{C}$ be the category of complexes of R-modules. Show that $\mathcal{C}$ is an abelian category.

Problem 6:
Let R be any ring. Show that the intersection of all maximal left ideals in R is equal to the intersection of all maximal right ideals in R.

Problem 7:
Prove that the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}) : \mathbb{Z}\text{-}\mathbf{mod}^{\text{op}} \to \mathbb{Z}\text{-}\mathbf{mod}$ is exact. (This means that $\mathbb{Q}$ is an *injective* $\mathbb{Z}$-module.)

Problem 7:
Let R be a commutative ring and $S \subset R$ a multiplicative subset.
(a) Show that $S^{-1}M = M \otimes_R S^{-1}R$ for any R-module M.
(b) The functor $F : R\text{-}\mathbf{mod} \to S^{-1}R\text{-}\mathbf{mod}$ defined by $F(M) = S^{-1}M$ is exact.

Problem 8:

Let R be a commutative local Noetherian ring with residue field $k = R/\mathfrak{m}$, and let M be a finitely generated R-module. Choose $x_1, \ldots, x_n \in M$ such that $\overline{x_1}, \ldots, \overline{x_n}$ is a basis for the k-vector space $M/\mathfrak{m}M$.

(a) Prove that M is generated by $x_1, \ldots, x_n$ as an R-module.

(b) If M is flat, then M is a free R-module with basis $x_1, \ldots, x_n$.

Hint: You may need the fact that any submodule of a finitely generated module over a Noetherian ring is finitely generated.

Problem 9:

Let R be a commutative Noetherian ring and M a finitely generated R-module. Then the following are equivalent:

(a) M is a projective R-module.

(b) M is a flat R-module.

(c) For every prime ideal $P \subset R$, M_P is a free R_P-module.