Algebra 2, Homework 2 Solutions

Problem 1:

Let $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ where k is an algebraically closed field. Show that W is Zariski closed and find $I(W) \subset k[x, y, z]$.

Set $J = \langle x^2 - y^3, y^2 - z^3\rangle \subset k[x, y, z]$. Then $W = Z(J)$ is Zariski closed and $J \subset I(W)$. The ring $R = k[x, y, z]/J$ is a free module over the subring $k[z]$ with basis $1, x, y, xy$. It follows that any polynomial $f \in k[x, y, z]$ can be written as $f = g_1(z) + g_2(z)x + g_3(z)y + g_4(z)xy + h$ where $g_i \in k[z]$ and $h \in J$. Notice that for $t \in k$ we have $f(t^9, t^6, t^4) = g_1(t^4) + g_2(t^4)t^9 + g_3(t^4)t^6 + g_4(t^4)t^{15}$. If $f \in I(W)$, then $f(t^9, t^6, t^4) = 0$ for all $t \in k$, so $g_1(T^4) + g_2(T^4)T^9 + g_3(T^4)T^6 + g_4(T^4)T^{15} = 0 \in k[T]$. Since $1, T^9, T^6, T^{15} \in k[T]$ are linearly independent over the subring $k[T^4]$, we deduce that $g_1 = g_2 = g_3 = g_4 = 0$, so $f = h \in J$. This shows that $I(W) = J = \langle x^2 - y^3, y^2 - z^3\rangle$.

Problem 2:

Set $I = (y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3) \subset k[x, y]$, where k is an algebraically closed field. Find the radical $\sqrt{I} \subset k[x, y]$. (Does it depend on the characteristic of k?)

If $\mathrm{char}(k) = 3$ then $\sqrt{I} = I(Z(I)) = I(\{(0, 0)\} \cup (\{1\} \times k)) = \langle x^2 - x, xy - y\rangle$.

If $\mathrm{char}(k) \neq 3$ then $\sqrt{I} = I(Z(I)) = I(\{(0, 0), (1, 0)\}) = \langle x^2 - x, y\rangle$.

Problem 3:

(a) Let K be a field and $R, S \subset K$ two normal subrings. Show that $R \cap S$ is also a normal ring.

Let $R_0 = (R \smallsetminus 0)^{-1}R \subset K$ denote the field of fractions of R. Notice that $(R \cap S)_0 \subset R_0 \cap S_0 \subset K$. By assumption R is integrally closed in R_0 and S is integrally closed in S_0. Let $x \in (R \cap S)_0$ be integral over $R \cap S$. Then $x \in R_0$ is integral over R, so $x \in R$. Similarly, $x \in S_0$ is integral over S, so $x \in S$. We obtain $x \in R \cap S$. This shows that $R \cap S$ is a normal ring.

(b) Show that if k is any field, then $R = k[x, y, z]/\langle z^2 - xy\rangle$ is a normal ring. Hint: $R \cong k[x, xt, xt^2] \subset k(x, t)$.

Let $\phi : k[x, y, z] \to k[x, t]$ be the ring homomorphism defined by $\phi(x) = x$, $\phi(y) = xt^2$, and $\phi(z) = xt$. Then $\langle z^2 - xy\rangle \subset \mathrm{Ker}(\phi)$. The ring R is a free $k[x, y]$-module with basis $1, z$, so every polynomial $f(x, y, z) \in k[x, y, z]$ can be written as $f(x, y, z) = f_0(x, y) + f_1(x, y)z + h$ where $f_0, f_1 \in k[x, y]$ and $h \in \langle z^2 - xy\rangle$. If $f \in \mathrm{Ker}(\phi)$, then $f_0(x, xt^2) + f_1(x, xt^2)xt = 0 \in k[x, t]$. Since 1 and xt are linearly independent over the subring $k[x, t^2]$, it follows that $f_0 = f_1 = 0 \in k[x, y]$, so $f = h \in \langle z^2 - xy\rangle$. This shows that $\mathrm{Ker}(\phi) = \langle z^2 - xy\rangle$, so $R \cong k[x, xt, xt^2] \subset k(x, t)$. Notice that $k[x, xt, xt^2] = k[x, t] \cap k[xt^2, t^{-1}] \subset k(x, t)$. Since $k[x, t]$ and $k[xt^2, t^{-1}]$ are normal rings, we deduce that $R \cong k[x, xt, xt^2]$ is normal.

Problem 4:

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. Prove that $F(0) = 0$ and $F(A \oplus B) = F(A) \oplus F(B)$ for all objects $A, B \in \mathrm{ob}(\mathcal{C})$.

Set $Z = F(0_{\mathcal{C}})$, let $0_Z \in \mathrm{Hom}_{\mathcal{D}}(Z, Z)$ be the zero morphism, and let $1_Z \in \mathrm{Hom}_{\mathcal{D}}(Z, Z)$ be the identity morphism. Then $1_Z = F(1_0) = F(0_0) = 0_Z$, the last equality because F is additive. If $M \in \mathrm{ob}\,\mathcal{D}$ is any object and $f \in \mathrm{Hom}_{\mathcal{D}}(Z, M)$, then $f = f \circ 1_Z = f \circ 0_Z = 0_{Z,M}$, so $\mathrm{Hom}_{\mathcal{D}}(Z, M) = \{0_{Z,M}\}$. Similarly we obtain $\mathrm{Hom}_{\mathcal{D}}(M, Z) = \{0_{M,Z}\}$. It follows that Z is a zero object in $\mathcal{D}$.

Let $A_1, A_2 \in \operatorname{ob} \mathcal{C}$ and set $S = A_1 \oplus A_2$. Then there are morphisms $p_j : S \to A_j$ and $i_j : A_j \to S$ for $j = 1, 2$, such that $p_j i_k = \delta_{jk} 1_{A_j}$ and $i_1 p_1 + i_2 p_2 = 1_S$. Set $I_j = F(i_j) : F(A_j) \to F(S)$ and $P_j = F(p_j) : F(S) \to F(A_j)$. Since F is additive we then obtain $P_j I_k = \delta_{jk} 1_{F(A_j)}$ and $I_1 P_1 + I_2 P_2 = 1_{F(S)}$. This shows that $F(A_1 \oplus A_2) = F(S) = F(A_1) \oplus F(A_2)$.

Problem 5:

Let R be a ring and let $\mathcal{C}$ be the category of complexes of R-modules. Show that $\mathcal{C}$ is an abelian category.

Given a morphism $\alpha : A_* \to B_*$ of complexes of R-modules, define $K_i = \operatorname{Ker}(\alpha_i : A_i \to B_i)$ and $C_i = \operatorname{Coker}(\alpha_i)$. The differentials of A_* and B_* define differentials $K_i \to K_{i-1}$ and $C_i \to C_{i-1}$. We obtain new complexes K_* and C_*, and the inclusion $K_* \to A_*$ is a kernel of α while the projection $B_* \to C_*$ is a cokernel. The other axioms of an abelian category should also be checked. (This was a bit short, but nothing is difficult.)

Problem 6:

Let R be any ring. Show that the intersection of all maximal left ideals in R is equal to the intersection of all maximal right ideals in R.

This is a consequence of the results proved in chapter 4. Here is a summary of the ideas. Define $\operatorname{rad}(R) \subset R$ and $\operatorname{rad}'(R) \subset R$ by

$$\operatorname{rad}(R) = \bigcap_{I \subset R \text{ max left}} I \qquad \text{and} \qquad \operatorname{rad}'(R) = \bigcap_{J \subset R \text{ max right}} J \,.$$

Then $\operatorname{rad}(R)$ is a left ideal of R and $\operatorname{rad}'(R)$ is a right ideal.

We first show that $\operatorname{rad}(R)$ is a two-sided ideal. This follows from the identity

$$\operatorname{rad}(R) = \bigcap_{I \subset R \text{ max left}} \operatorname{ann}_R(R/I) \,.$$

One inclusion is clear because $\operatorname{ann}_R(R/I) \subset I$. The other inclusion follows because

$$\operatorname{ann}_R(R/I) = \bigcap_{0 \neq y \in R/I} \operatorname{ann}_R(y) \,.$$

Notice that if I is a maximal left ideal in R, then R/I is an irreducible left R-module, so for $0 \neq y \in R/I$ we have $R/\operatorname{ann}_R(y) \cong Ry = R/I$, which shows that $\operatorname{ann}_R(y)$ is a maximal left ideal of R.

We next show that every element of $\operatorname{rad}(R)$ is quasi-regular. Let $z \in \operatorname{rad}(R)$. If z is not left quasi-regular, then $R(1-z) \neq R$, so we can choose a maximal left ideal I such that $R(1-z) \subset I \subsetneq R$. But then $z, 1-z \in I$, so $1 \in I$, a contradiction. Since z is left quasi-regular, we can choose $z' \in R$ such that $(1-z')(1-z) = 1$. Then $z' = z'z - z \in \operatorname{rad}(R)$, so z' is also left quasi-regular, and we can find $z'' \in R$ such that $(1-z'')(1-z') = 1$. We obtain $z = z''$ and $(1-z)(1-z') = (1-z')(1-z) = 1$.

Assume that $\operatorname{rad}(R) \not\subset \operatorname{rad}'(R)$. Then we can choose a maximal right ideal J such that $\operatorname{rad}(R) \not\subset J$. Since $\operatorname{rad}(R)$ is a right ideal, we obtain $\operatorname{rad}(R) + J = R$, so we can write $1 = z + b$ with $z \in \operatorname{rad}(R)$ and $b \in J$. Since z is quasi-regular, we can choose $z' \in R$ such that $b(1-z') = (1-z)(1-z') = 1$. But then $1 \in J$, a contradiction. We deduce that $\operatorname{rad}(R) \subset \operatorname{rad}'(R)$. A symmetric argument gives $\operatorname{rad}'(R) \subset \operatorname{rad}(R)$.

Problem 7:

Prove that the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}) : \mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{op}} \to \mathbb{Z}\text{-}\mathbf{mod}$ is exact. (This means that $\mathbb{Q}$ is an *injective* $\mathbb{Z}$-module.)

Let $A \subset B$ be $\mathbb{Z}$-modules. It is enough to show that the restriction map $\mathrm{Hom}_{\mathbb{Z}}(B,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q})$ is surjective. Let $\phi \in \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q})$ and consider the set S of pairs (C,ψ) for which C is a submodule with $A \subset C \subset B$, $\psi : C \to \mathbb{Q}$ is a $\mathbb{Z}$-homomorphism, and $\psi|_A = \phi$. Define a partial order on S by $(C,\psi) \leq (C',\psi')$ if and only if $C \subset C'$ and $\psi'|_C = \psi$. Then S is inductively ordered, so by Zorn's lemma we can choose a maximal element $(C,\psi) \in S$. We claim that $C = B$. Otherwise choose $x \in B \smallsetminus C$ and set $C' = C + \mathbb{Z}x$. If $C \cap \mathbb{Z}x = 0$, then set $\psi' = \psi \oplus 0 : C' = C \oplus \mathbb{Z}x \to \mathbb{Q}$. We obtain $(C,\psi) < (C',\psi')$, a contradiction. Otherwise we have $C \cap \mathbb{Z}x = \mathbb{Z}mx$ for some $m \geq 2$. This time we obtain a well-defined extension $\psi' : C' \to \mathbb{Q}$ of ψ by setting $\psi'(c+px) = \psi(c) + \frac{p}{m}\psi(mx)$ for $c \in C$ and $p \in \mathbb{Z}$. We obtain $(C,\psi) < (C',\psi')$, a contradiction. We conclude that $C = B$. Now $\phi \in \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q})$ is the image of $\psi \in \mathrm{Hom}_{\mathbb{Z}}(B,\mathbb{Q})$, as required.

Problem 7:

Let R be a commutative ring and $S \subset R$ a multiplicative subset.

(a) Show that $S^{-1}M = M \otimes_R S^{-1}R$ for any R-module M.

By localizing the R-homomorphism $M \to M \otimes_R S^{-1}R$ defined by $m \mapsto m \otimes 1$, we obtain an $S^{-1}R$-homomorphism $S^{-1}M \to M \otimes_R S^{-1}R$ given by $m/s \mapsto m \otimes 1/s$. The R-bilinear map $M \times S^{-1}R \to S^{-1}M$ defined by $m \otimes r/s \mapsto rm/s$ induces the inverse map.

(b) The functor $F : R\text{-}\mathbf{mod} \to S^{-1}R\text{-}\mathbf{mod}$ defined by $F(M) = S^{-1}M$ is exact.

Let $A \subset B$ be R-modules. We must show that the induced map $S^{-1}A \to S^{-1}B$ of $S^{-1}R$-modules is injective. Assume that $a/s \in S^{-1}A$ is mapped to zero in $S^{-1}B$. Then there exists $t \in S$ such that $ta = 0 \in B$. But then $ta = 0 \in A \subset B$, so $a/s = 0 \in S^{-1}A$.

Problem 8:

Let R be a commutative local Noetherian ring with residue field $k = R/\mathfrak{m}$, and let M be a finitely generated R-module. Choose $x_1, \dots, x_n \in M$ such that $\overline{x_1}, \dots, \overline{x_n}$ is a basis for the k-vector space $M/\mathfrak{m}M$.

(a) Prove that M is generated by $x_1, \dots, x_n$ as an R-module.

Set $N = \langle x_1, \dots, x_n \rangle \subset M$ and set $Q = M/N$. Then $0 \to N \to M \to Q \to 0$ is an exact sequence of R-modules. Since the functor $- \otimes_R k$ is right exact, we obtain an exact sequence of k-vector spaces $N/\mathfrak{m}N \to M/\mathfrak{m}M \to Q/\mathfrak{m}Q \to 0$. Since the first map is an isomorphism, we have $Q = \mathfrak{m}Q$, so $Q = 0$ by Nakayama's lemma.

(b) If M is flat, then M is a free R-module with basis $x_1, \dots, x_n$.

Consider the map $\phi : R^n \to M$ defined by $\phi(a_1, \dots, a_n) = a_1x_1 + \cdots + a_nx_n$, and set $K = \mathrm{Ker}(\phi) \subset R^n$. The short exact sequence $0 \to K \to R^n \to M \to 0$ gives the long exact sequence $\mathrm{Tor}_1^R(M,k) \to K/\mathfrak{m}K \to (R/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0$. The last map is an isomorphism, so if M is flat then it follows that $\mathrm{Tor}_1^R(M,k) = 0$ and $K/\mathfrak{m}K = 0$. Since R is Noetherian and R^n is a finitely generated R-module, we deduce that K is finitely generated, so $K = 0$ by Nakayama's lemma.

Problem 9:

Let R be a commutative Noetherian ring and M a finitely generated R-module. Then the following are equivalent:

(a) M is a projective R-module.
(b) M is a flat R-module.
(c) For every prime ideal $P \subset R$, M_P is a free R_P-module.

(a) $\Rightarrow$ (b): Any projective module is a direct summand of a free module.

(b) $\Rightarrow$ (c): Assume that M is a flat R-module and let $P \subset R$ be a prime ideal. It is enough to show that M_P is a flat R_P-module. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of R_P-modules. Then A_* is also an exact sequence of R-modules, and we have $M_P \otimes_{R_P} A_* = (M \otimes_R R_P) \otimes_{R_P} A_* = M \otimes_R (R_P \otimes_{R_P} A_*) = M \otimes_R A_*$. This sequence is exact because M is a flat R-module.

(c) $\Rightarrow$ (a): It is enough to show that if M_P is a projective R_P-module for every prime ideal $P \subset R$, then M is a projective R-module. For this we need to show that if $A \to B$ is a surjective homomorphism of R-modules, then $\operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B)$ is also surjective. Let C be the cokernel, so that $\operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B) \to C \to 0$ is an exact sequence of R-modules. We will show that $C = 0$. It follows from the lemma below that $\operatorname{Hom}_{R_P}(M_P, A_P) \to \operatorname{Hom}_{R_P}(M_P, B_P) \to C_P \to 0$ is exact for every prime ideal $P \subset R$. Since M_P is a projective R_P-module, this implies that $C_P = 0$. Assume that $0 \neq x \in C$ is any non-zero element. Then $\operatorname{ann}_R(x) \subsetneq R$ is a proper ideal, so we can find a maximal ideal P with $\operatorname{ann}_R(x) \subset P \subsetneq R$. But then $x/1 \neq 0 \in C_P$, a contradiction.

Lemma: Let R be a commutative ring, let M and N be R-modules, and let $S \subset R$ be a multiplicatively closed subset. Assume that M is finitely presented. Then $S^{-1} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$.

Proof: By localizing the map $\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ we obtain an $S^{-1}R$-homomorphism $S^{-1} \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$. If $M = R^k$ is a free of finite rank, then this map is an isomorphism since

$$\begin{aligned} S^{-1} \operatorname{Hom}_R(R^k, N) = S^{-1}(N^k) = (S^{-1}N)^k &= \operatorname{Hom}_{S^{-1}R}((S^{-1}R)^k, S^{-1}N) \\ &= \operatorname{Hom}_{S^{-1}R}(S^{-1}(R^k), S^{-1}N)\,. \end{aligned}$$

Suppose that M is finitely presented, and choose a presentation $F' \to F \to M \to 0$ where F' and F are finitely generated free R-modules. We have a commutative diagram with exact columns:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ S^{-1} \operatorname{Hom}_R(M, N) & \longrightarrow & \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \\ \downarrow & & \downarrow \\ S^{-1} \operatorname{Hom}_R(F, N) & \longrightarrow & \operatorname{Hom}_{S^{-1}R}(S^{-1}F, S^{-1}N) \\ \downarrow & & \downarrow \\ S^{-1} \operatorname{Hom}_R(F', N) & \longrightarrow & \operatorname{Hom}_{S^{-1}R}(S^{-1}F', S^{-1}N) \end{array}$$

Since the two lower horizontal maps are isomorphisms, so is the top horizontal map, as required.