One Semester of Algebraic Varieties

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Preface

This document contains class notes from a course on Algebraic Varieties taught by Anders Buch at Rutgers in the fall of 2006. The notes were typeset in TeX by Charles Siegel in real time from the blackboard, undoubtedly fixing many mistakes in the process. Any that remain should be blamed on the lecturer.

The content of the notes is roughly equivalent to courses that were taught earlier at Massachusetts Institute of Technology (in the fall of 2000 and in the fall of 2001) and at Aarhus University (in the spring of 2003). We owe many thanks to the participants in these courses.

The choice of content is inspired or stolen from various sources, including Kempf's book *Algebraic Varieties*, Hartshornes's book *Algebraic Geometry*, and notes from an earlier course at MIT by Ravi Vakil. The main references on commutative algebra is Eisenbud's book *Commutative Algebra with a view toward Algebraic Geometry* and Lang's book *Algebra*.

Chapter 1

Affine Varieties

We will begin following Kempf's Algebraic Varieties, and eventually will do things more like in Hartshorne. We will also use various sources for commutative algebra.

What is algebraic geometry? Classically, it is the study of the zero sets of polynomials.

We will now fix some notation. k will be some fixed algebraically closed field, any ring is commutative with identity, ring homomorphisms preserve identity, and a k-algebra is a ring R which contains k (i.e., we have a ring homomorphism $\iota : k \to R$).

$P \subseteq R$ an ideal is prime iff R/P is an integral domain.

1.1 Algebraic Sets

We define affine n-space, $\mathbb{A}^n = k^n = \{(a_1, \ldots, a_n) : a_i \in k\}$.

Any $f = f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ defines a function $f : \mathbb{A}^n \to k : (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$.

Exercise If $f, g \in k[x_1, \ldots, x_n]$ define the same function then $f = g$ as polynomials.

Definition 1 (Algebraic Sets)**.** *Let $S \subseteq k[x_1, \ldots, x_n]$ be any subset. Then $V(S) = \{a \in \mathbb{A}^n : f(a) = 0$ for all $f \in S\}$.*

A subset of $\mathbb{A}^n$ is called algebraic if it is of this form.

e.g., a point $\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$.

Exercises

1. $I = (S)$ is the ideal generated by S. Then $V(S) = V(I)$.

2. $I \subseteq J \Rightarrow V(J) \subseteq V(I)$.

3. $V(\cup_\alpha I_\alpha) = V(\sum I_\alpha) = \cap V(I_\alpha)$.

4. $V(I \cap J) = V(I \cdot J) = V(I) \cup V(J)$.

Definition 2 (Zariski Topology)**.** *We can define a topology on $\mathbb{A}^n$ by defining the closed subsets to be the algebraic subsets. $U \subseteq \mathbb{A}^n$ is open iff $\mathbb{A}^n \setminus U = V(S)$ for some $S \subseteq k[x_1, \ldots, x_n]$.*

Exercises 3 and 4 imply that this is a topology.

The closed subsets of $\mathbb{A}^1$ are the finite subsets and $\mathbb{A}^1$ itself.

Definition 3 (Ideal of a Subset)**.** *If $W \subset \mathbb{A}^n$ is any subset, then $I(W) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in W\}$*

Facts/Exercises

1. $V \subseteq W \Rightarrow I(W) \subseteq I(V)$

2. $I(\emptyset) = (1) = k[x_1, \ldots, x_n]$

3. $I(\mathbb{A}^n) = (0)$.

Definition 4 (Affine Coordinate Ring)**.** *$W \subset \mathbb{A}^n$ is algebraic. Then $A(W) = k[W] = k[x_1, \ldots, x_n]/I(W)$*

We can think of this as the ring of all polynomial functions $f : W \to k$.

Definition 5 (Radical Ideal)**.** *Let R be a ring and $I \subseteq R$ be an ideal, then the radical of I is the ideal $\sqrt{I} = \{f \in R : f^i \in I \text{ for some } i \in \mathbb{N}\}$*

We call I a radical ideal if $I = \sqrt{I}$.

Exercise

If I is an ideal, then $\sqrt{I}$ is a radical ideal.

Proposition 1. *$W \subseteq \mathbb{A}^n$ any subset, then $I(W)$ is a radical ideal.*

Proof. We have that $I(W) \subseteq \sqrt{I(W)}$.

Suppose $f \in \sqrt{I(W)}$. Then $f^i \in I$ for some i. That is, for all $a \in W$, $f^i(a) = 0$. Thus, $f(a)^m = 0 = f(a)$. And so, $f(a) \in I$. □

Exercises

1. $S \subseteq k[x_1, \ldots, x_n]$, then $S \subseteq I(V(S))$.
2. $W \subseteq \mathbb{A}^n$ then $W \subseteq V(I(W))$.
3. $W \subseteq \mathbb{A}^n$ is an algebraic subset, then $W = V(I(W))$.
4. $I \subseteq k[x_1, \ldots, x_n]$ is any ideal, then $V(I) = V(\sqrt{I})$ and $\sqrt{I} \subseteq I(V(I))$

Theorem 1 (Nullstellensatz)**.** *Let k be an algebraically closed field, and $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then $\sqrt{I} = I(V(I))$.*

Corollary 1. *$k[V(I)] = k[x_1, \ldots, x_n]/\sqrt{I}$.*

To prove the Nullstellensatz, we will need the following:

Theorem 2 (Nöther's Normalization Theorem)**.** *If R is any finitely generated k-algebra (k can be any field), then there exist $y_1, \ldots, y_m \in R$ such that $y_1, \ldots, y_m$ are algebraically independent over k and R is an integral extension of the subring $k[y_1, \ldots, y_m]$.*

Proof is in Eisenbud and other Commutative Algebra texts.

Theorem 3 (Weak Nullstellensatz)**.** *Let k be an algebraically closed field, and $I \subsetneq k[x_1, \ldots, x_n]$ any proper ideal, then $V(I) \neq \emptyset$.*

Proof. We may assume without loss of generality that I is actually a maximal ideal. Then $R = k[x_1, \ldots, x_n]/I$ is a field. R is also a finitely generated k-algebra, and so by Normalization, $\exists y_1, \ldots, y_m \in R$ such that $y_1, \ldots, y_m$ are algebraically independent over k and that R is integral over $k[y_1, \ldots, y_m]$.

Claim: $m = 0$. Otherwise, $y_1^{-1} \in R$ is integral over $k[y_1, \ldots, y_m]$, and so then $y_1^{-p} + y_1^{1-p} f_1 + \ldots + y_1^{-1} f_{p-1} + f_p = 0$ for $f_i \in k[y_1, \ldots, y_m]$. Multiplying through by y_1^p gives $1 = -(y_1 f_1 + \ldots + y_1^{p-1} f_{p-1} + y_1^p f_p) \in (y_1)$, which contradicts the algebraic independence.

Thus, the field R is algebraic over k. As k is algebraically closed, $R = k$. $k \subseteq k[x_1, \ldots, x_n] \to R = k$

Let a_i = the image in k of x_i. Then $x_i - a_i \in I$. Thus, the ideal generated by $(x_1 - a_1, \ldots, x_n - a_n) \subseteq I \subsetneq k[x_1, \ldots, x_n]$, and so they $I = (x_1 - a_1, \ldots, x_n - a_n)$, as it is a maximal ideal.

$V(I) = V(x_1 - a_1, \ldots, x_n - a_n) = \{(a_1, \ldots, a_n)\} \neq \emptyset$ $\square$

Note: Any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with $a_i \in k$.

This is NOT true over $\mathbb{R}$, look at the ideal $(x^2+1) \subseteq \mathbb{R}[x]$. It is, in fact, maximal.

Now, we can prove the Nullstellensatz.

Proof. Let $I \subseteq k[x_1, \ldots, x_n]$ be any ideal. We will prove that $I(V(I)) = \sqrt{I}$.

It was an exercise that $\sqrt{I} \subseteq I(V(I))$.

Let $f \in I(V(I))$. We must show that $f \in \sqrt{I}$.

Looking at $\mathbb{A}^{n+1}$, we have the variables, $x_1, \ldots, x_n, y$. Set $J = (I, 1 - yf) \subseteq k[x_1, \ldots, x_n, y]$.

Claim: $V(J) = \emptyset \subset \mathbb{A}^{n+1}$. This is as, if $p = (a_1, \ldots, a_n, p) \in V(J)$, then $(a_1, \ldots, a_n) \in V(I)$, then $(1 - yf)(p) = 1 - bf(a_1, \ldots, a_n)$. But $f(a_1, \ldots, a_n) = 0$, so $(1 - yf)(p) = 1$, and so $p \notin V(J)$.

By the Weak Nullstellensatz, $J = k[x_1, \ldots, x_n, y]$. Thus $1 = h_1 g_1 + \ldots + h_m g_m + q(1 - yf)$ where $g_1, \ldots, g_m \in I$ and $h_1, \ldots, h_m, q \in k[x_1, \ldots, x_n, y]$.

Set $y = f^{-1}$, and multiply by some big power of f to get a polynomial equation once more.

Then $f^N = \tilde{h}_1 g_1 + \ldots + \tilde{h}_m g_m$ where the $\tilde{h}_i = f^N h_i(x_1, \ldots, x_n)$.

And so, we have $f^N \in I$, and thus, $f \in \sqrt{I}$, by definition. $\square$

<u>exercise</u>: $V(I(W)) = \overline{W}$ in the Zariski Topology.

1.2 Irreducible Algebraic Sets

Recall: $V(y^2 - xy - x^2 y + x^3) = V(y - x) \cup V(y - x^2)$, and $V(xz, yz) = V(x, y) \cup V(z)$.

Definition 6 (Reducible Subsets)**.** *A Zariski Closed subset $W \subseteq \mathbb{A}^n$ is called reducible if $W = W_1 \cup W_2$ where $W_i \subsetneq W$ and W_i closed.*

Otherwise, we say that W is irreducible.

Proposition 2. *Let $W \subseteq \mathbb{A}^n$ be closed. Then W is irreducible iff $I(W)$ is a prime ideal.*

Proof. $\Rightarrow$: Suppose $I(W)$ is not prime. Then $\exists f_1, f_2 \notin I(W)$ such that $f_1 f_2 \in W$. Set $W_1 = W \cap V(f_1)$ and $W_2 = W \cap V(f_2)$.

As $f_i \notin I(W)$, $W \not\subseteq V(f_i)$, and so $W_i \subsetneq W$. Now we must show that $W = W_1 \cup W_2$. Let $a \in W$. Assume $a \notin W_1$. Then $f_1(a) \neq 0$, but $f_1(a) f_2(a) = 0$, so $f_2(a) = 0$, thus $a \in W_2$.

$\Leftarrow$: Exercise $\square$

This gives us the beginning of an algebra-geometry dictionary.

Algebra	Geometry
$k[x_1, \ldots, x_n]$	$\mathbb{A}^n$
radical ideals	closed subsets
prime ideals	irreducible closed subsets
maximal ideals	points

In fact, this is an order reversing correspondence. So $I \subseteq J \iff V(I) \supseteq V(J)$, but this requires I, J to be radical.

Definition 7 (Nötherian Ring). *A ring R is called Nötherian if every ideal $I \subseteq R$ is finitely generated.*

Exercise A ring R is Nötherian iff every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ stabilizes, that is, $\exists N$ such that $I_N = I_{N+1} = \ldots$.

Theorem 4 (Hilbert's Basis Theorem). *If R is Nötherian, then $R[x]$ is Nötherian.*

Corollary 2. *$k[x_1, \ldots, x_n]$ is Nötherian.*

Definition 8 (Nötherian Topological Space). *A topological space X is Nötherian if every descending chain of closed subsets stabilizes.*

Corollary 3. *$\mathbb{A}^n$ is Nötherian.*

$W_1 \supseteq W_2 \supseteq \ldots$ closed in $\mathbb{A}^n$, then $I(W_1) \subseteq I(W_2) \subseteq \ldots$ ideals in $k[x_1, \ldots, x_n]$, and so must stabilize.

Theorem 5. *Any closed subset of a Nötherian Topological Space X is a union of finitely many irreducible closed subsets.*

Proof. Assume the result is false. $\exists W$ a closed subset of X which is not the union of finitely many irreducible closed sets.

As X is Nötherian, we may assume that W is a minimal counterexample. W is not irreducible, and so $W = W_1 \cup W_2$, where $W_i \subsetneq W$ and W_i closed. The W_i can't be counterexamples, as W is a minimal one, but then $W = W_1 \cup W_2$ and each W_i is the union of finitely many irreducible closed sets. Thus, W cannot be a counterexample. $\square$

Corollary 4. *Every closed $W \subseteq \mathbb{A}^n$ is union of finitely many irreducible closed subsets.*

Example: $V(xy) = V(x) \cup V(y) \cup V(x-1, y)$.

Recall: X is a topological space, then if $Y \subseteq X$ is any subset, it has the subspace topology, that is, $U \subseteq Y$ is open iff $\exists U' \subseteq X$ open such that $U = U' \cap Y$.

Note:

1. $W \subseteq Y$ is closed iff $W = \overline{W} \cap Y$, where the closure is in X.

2. X is Nötherian implies that Y is Nötherian in the subspace topology.

Definition 9 (Zariski Topology on $X \subseteq \mathbb{A}^n$)**.** *If $X \subseteq \mathbb{A}^n$, then the Zariski Topology on X is the subspace topology.*

Definition 10 (Components of X)**.** *If X is any Nötherian Topological Space, then the maximal irreducible closed subsets of X are called the (irreducible) components of X.*

Exercises

1. X has finitely many components.

2. $X =$ the union of its irreducible components.

3. $X \neq$ union of any proper subset of its components.

4. A topological space is Nötherian if and only if every subset is quasi-compact.

5. A Nötherian Hausdorff space is finite.

Recall: $X \subseteq \mathbb{A}^n$ closed. Then $A(X) = k[x_1, \ldots, x_n]/I(X)$.

Definition 11. *If $f \in A(X)$, set $D(f) = \{a \in X : f(a) \neq 0\}$.*

Proposition 3. *The sets $D(f)$ form a basis for the Zariski Topology on X.*

Proof. Let $p \in U \subseteq X$, U open. Show that $p \in D(f) \subseteq U$ for some $f \in A(X)$. $Z = X \setminus U$ a closed subset of X, and $Z \subsetneq Z \cup \{p\}$ implies that $I(Z) \supsetneq I(Z \cup \{p\})$.

Take any $f \in I(Z) \setminus I(Z \cup \{p\})$. Then f vanishes on Z but not at p, so $p \in D(f)$. $\square$

1.3 Regular Functions

Let $X \subseteq \mathbb{A}^n$ be an algebraic subset, and $U \subseteq X$ is a relatively open subset of X.

Definition 12 (Regular Function). *A function $f : U \to k$ is called regular if f is locally rational. That is, $\exists$ open cover $U = \cup_\alpha U_\alpha$ and functions $p_\alpha, q_\alpha \in A(X)$ such that $\forall a \in U_\alpha$, $q_\alpha(a) \neq 0$ and $f(a) = p_\alpha(a)/q_\alpha(a)$.*

We define $k[U]$ to be the set of regular functions from U to k.

Note:

1. $k[U]$ is a k-algebra.
2. $A(X) \subseteq k[X]$.

Example

Let $X = V(xy - zw) \subseteq \mathbb{A}^4$. $f : U \to k$ can be defined by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Thus, $f \in k[U]$.

Exercise; $\nexists p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Lemma 1. *Let $q_1, \ldots, q_n \in A(X)$. Then $D(q_1) \cup \ldots \cup D(q_n) = X$ iff $(q_1, \ldots, q_m) = (1) = A(X)$.*

Proof. $\Leftarrow$: $1 = \sum h_i q_i$, $h_i \in A(X)$, then the q_i cannot all vanish at any point, and so we are done.

$\Rightarrow$: Take $Q_i \in k[x_1, \ldots, x_n]$ such that $q_i = \overline{Q_i} \in A(X)$. $D(q_1) \cup \ldots \cup D(q_m) = X$, so $X \cap V(Q_1) \cap \ldots \cap V(Q_m) = \emptyset = V(I(X), Q_1, \ldots, Q_m) = \emptyset$, and so, by the weak nullstellensatz, $(I(X), Q_1, \ldots, Q_m) = (1) \subseteq k[x_1, \ldots, x_n]$, and so $(q_1, \ldots, q_m) = (1) = A(X)$. □

Theorem 6. *Let $X \subseteq \mathbb{A}^n$ be an algebraic set. Then $k[X] = A(X)$.*

Proof. Let $f \in k[X]$. Then $X = U_1 \cup \ldots \cup U_m$ and there are $p_i, q_i \in A(X)$ such that $q_i \neq 0$ and $f = p_i/q_i$ on U_i.

We can refine the open cover such that each $U_i = D(g_i)$ for some g_i. Note: $f = p_i/q_i = \frac{p_i g_i}{q_i g_i}$ on $U_i = D(g_i) = D(g_i q_i)$. We can replace p_i with $p_i g_i$ and q_i by $q_i g_i$.

Then we can assume that U_i is $D(q_i) = D(q_i^2)$. Thus, $X = D(q_1^2) \cup \ldots \cup D(q_m^2)$. By the lemma, we know that $1 = \sum_{i=1}^m h_i q_i^2$, $h_i \in A(X)$. Note $q_i^2 f = q_i p_i$ on U_i, and $q_i = 0$ outside of U_i.

$f = 1f = \sum_{i=1}^m h_i q_i^2 f = \sum_{i=1}^m h_i q_i p_i$, so $f \in A(X)$. □

1.4 Spaces with functions

Definition 13. *A space with functions (SWF) is a topological space X together with an assignment to each open $U \subseteq X$ of a $k-$algebra $k[U]$ consisting of functions $U \to k$. These are called regular functions. It must also satisfy the following:*

1. *If $U = \cup U_\alpha$ is an open cover and $f : U \to k$ any function, then f is regular on U iff $f|_{U_\alpha}$ is regular on U_α for all α.*

2. *If $U \subseteq X$ is open, $f \in k[U]$, then $D(f) = \{a \in U : f(a) \neq 0\}$ is open and $\frac{1}{f} \in k[D(f)]$.*

Note: $\mathscr{O}_X(U) = k[U]$ is another common notation.
Examples:

1. Algebraic sets. These are called Affine Algebraic Varieties

2. M is a differentiable manifold, $k = \mathbb{R}$, $k[U] = \{C^\infty \text{ functions } U \to \mathbb{R}\}$.

3. X is a SWF, $U \subseteq X$ open subset, then U is a SWF. $\mathscr{O}_U(V) = \mathscr{O}_X(V)$.

Definition 14 (Morphism of SWFs)**.** *Let X, Y be SWFs, then a morphism $\varphi : X \to Y$ is a continuous map which pulls back regular functions to regular functions. i.e., if $V \subseteq Y$ is open, $f \in \mathscr{O}_Y(V)$, then $\varphi^*(f) \in \mathscr{O}_X(\varphi^{-1}(V))$, $\varphi^*(f) = f \circ \varphi$.*

Definition 15 (Isomorphism)**.** *$\varphi : X \to Y$ is an isomorphism if φ is a morphism and $\exists$ a morphism $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$.*

Exercises

1. The id function of a SWF is a morphism.

2. Compositions of morphisms are morphisms.

3. Let X be any SWF and $Y \subseteq \mathbb{A}^n$ closed, that is, an affine variety. Then $f = (f_1, \ldots, f_n) : X \to Y$, is a morphism iff $f_i \in k[X]$ for all i.

Example: $\mathbb{A}^1 \setminus \{0\}$ is (isomorphic to) an affine variety defined by $V(1-xy)$.

1.5 Localization

Let R be a ring and $S \subseteq R$ multiplicatively closed subset. That is, $s, t \in S \Rightarrow st \in S$ and $1 \in S$.

We define $S^{-1}R = \{f/s : f \in R, s \in S\}$. We consider f/s to be the same element as g/t iff there exists $u \in S$ such that $u(ft - sg) = 0$. This is a ring by $\frac{f}{s}\frac{g}{t} = \frac{fg}{st}$ and $\frac{f}{s} + \frac{g}{t} = \frac{ft+gs}{st}$.

Exercise: Check these assertions.

Special Case: If $f \in R$, then $R_f = S^{-1}R$ where $S = \{f^n : n \in \mathbb{N}\}$. In fact, $R_f \cong R[y]/(1 - fy)$.

Definition 16 (Reduced Ring)**.** *R is a reduced ring iff $f^n = 0$ implies $f = 0$ for all $f \in R$. Equivalently, $(0) = \sqrt{(0)}$.*

Facts:

1. R reduced implies $S^{-1}R$ is reduced
2. R/I reduced iff $I = \sqrt{I}$

Proposition 4. *Let $X \subseteq \mathbb{A}^n$ be a closed affine variety and $f \in A(X)$. Then $D(f)$ if an affine variety, with affine coordinate ring $A(X)_f$.*

Proof. Let $I = I(X) \subseteq k[x_1, \ldots, x_n]$ define $J = (I, yf - 1) \subseteq k[x_1, \ldots, x_n, y]$.

Let $\phi : D(f) \to V(J) \subseteq \mathbb{A}^{n+1}$ by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, f(a_1, \ldots, a_n)^{-1})$. Note: ϕ is an isomorphism.

What remains is to compute the coordinate ring. $A(X)$ is reduced, and so $A(X)_f$ is reduced. $A(X)_f = k[x_1, \ldots, x_n, y]/J$, so J is a radical ideal, so $J = \sqrt{J} = I(V(J))$. Therefore, $k[D(f)] = k[V(J)] = k[x_1, \ldots, x_n, y]/J = A(X)_f$. □

Definition 17 (Prevariety)**.** *A prevariety is a space with functions X such that X has a finite open cover $X = U_1 \cup \ldots \cup U_m$ where U_i is an affine variety.*

Example: Any affine variety is a prevariety.

Exercise: Any prevariety is a Nötherian Topological Space

Example: An open subset of a prevariety is a prevariety. This follows from the previous proposition and the fact that principle open sets are a basis for the topology of any affine variety.

Proposition 5. *X is a space with functions, $Y \subseteq \mathbb{A}^n$ an affine variety, we have a 1-1 correspondence:*

$\{$*morphisms* $X \to Y\}$ $\iff$ $\{$*k-algebra homomorphism* $A(Y) \to k[X]\}$ *by* $\varphi \iff \varphi^*$

Proof. Note $\phi \mapsto \phi^*$ is a well defined map. Write $A(\mathbb{A}^n) = k[y_1, \ldots, y_n]$, then $I(Y) \subseteq k[y_1, \ldots, y_n]$. Then $\bar{y_i}$ is the image of y_i in $A(Y)$. Assume that $\alpha : A(Y) \to k[X]$ is a k-algebra homomorphism. We define $\phi : X \to \mathbb{A}^n$ by $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$.

If $f \in I(Y)$ then $f(\bar{y_1}, \ldots, \bar{y_n}) = 0$, so $f(\phi_1, \ldots, \phi_n) = \alpha(f(\bar{y_1}, \ldots, \bar{y_{n)}}) = 0$, and so $\phi(X) \subseteq Y$. Note, $\phi^*(\bar{y_i}) = y_i \circ \phi = \phi_i = \alpha(\bar{y_i})$. Thus, $\phi^* = \alpha$.

If $\phi : X \to Y$ is a morphism, then $\phi_i = y_i \circ \phi = \phi^*(\bar{y_i})$

Thus, ϕ is the morphism that we construct from ϕ^*. □

Corollary 5. *Two affine varieties are isomorphic iff their affine coordinate rings are isomorphic as k-algebras*

Exercise: $\mathbb{A}^n \setminus \{(0, \ldots, 0)\}$ is not affine for $n \geq 2$.

Proposition 6. *We have a one-to-one correspondence between affine varieties and reduced finitely generated k-algebras, up to isomorphism, by $X \mapsto k[X]$.*

Proof. Last time, we proved that two affine varieties are isomorphic iff their coordinate rings are isomorphic. Thus, $X \mapsto k[X]$ is injective.

Let R be a finitely generated reduced k-algebra generated by $r_1, \ldots, r_n \in R$. There is a k-alg homomorphism $\phi k[x_1, \ldots, x_n] \to R$ by $x_i \mapsto r_i$ which is surjective. Set $I = \ker \phi$ and let $X = V(I) \subseteq \mathbb{A}^n$.

I is radical, as R is reduced, so $k[X] = k[x_1, \ldots, x_n]/I \simeq R$. □

Note: Assume $m \subseteq R$ is a maximal ideal, then $\phi : k[x_1, \ldots, x_n] \to R$ as in proof, then $M = \phi^{-1}(m)$ is maximal, and $M = (x_1 - a_1, \ldots, x_n - a_n)$. $R/m = k[x_1, \ldots, x_n]/M = k$.

Canonical Construction

Let R be a finitely generated reduced k-algebra. Then define $\operatorname{Spec} -m(R) = \{m \subseteq R \text{ max ideals}\}$.

The topology will be that the closed sets $V(I) = \{m \supseteq I | I \subseteq R$ and ideal $\}$.

Let $f \in R$. We define $f : \operatorname{Spec} -m(R) \to k$ by $f(m) =$image of f in $R/m = k$. So f is a function from $\operatorname{Spec} -m$ to k.

I.E., $f(m) \in k \subseteq R$ is the unique element such that $f - f(m) \in m$.

Finally, if $U \subseteq \text{Spec} - m$ is open, $f : U \to k$ is some function, then f is regular if f is locally of the form $f(m) = p(m)/q(m)$ where $p, q \in R$.

Exercise: $\text{Spec} - m(R) \cong X$, where X is the affine variety with coordinate ring R, as spaces with functions.

Subspaces of SWFs

Let X be any space with functions, and $Y \subseteq X$ any subset. Then give Y an "inherited" SWF structure as follows:

We give Y the subspace topology, and if $U \subseteq Y$ is open and $f : U \to k$ is a function, then f is regular iff f can be locally extended to a regular function on X. That is, for every point $y \in U$, there is an open subset $U' \subseteq X$ containing y and $F \in \mathscr{O}_X(U')$ such that $f(x) = F(x)$ for all $x \in U \cap U'$.

Exercises

1. Y is a SWF
2. $i : Y \to X$ the inclusion map is a morphism.
3. Let Z be a SWF, $\phi : Z \to Y$ function. Then ϕ is a morphism iff $i \circ \phi$ is a morphism.
4. The SWF structure on Y is uniquely determined by (2) and (3) together.
5. Let $Z \subseteq Y \subseteq X$. Then Z inherits the same structure from Y and X.

Example: $X \subset \mathbb{A}^n$ an algebraic set inherits structure from $\mathbb{A}^n$. If $Y \subseteq X$ is closed, then Y inherits structure from X (or $\mathbb{A}^n$).

Proposition 7. *A closed subset of a prevariety is a prevariety.*

Proof. Let X be a prevariety, and $Y \subseteq X$ a closed subset. $X = U_1 \cup \ldots \cup U_m$ where U_i are open affine subsets of X.

$U_i \cap Y$ is a closed subset of U_i, which implies that $U_i \cap Y$ is affine, and so Y has the open cover $(U_1 \cap Y) \cup \ldots \cup (U_n \cap Y)$, and so is a prevariety. $\square$

Chapter 2

Projective Varieties

2.1 Projective space

Theorem 7. *Two distinct lines in the place intersect in exactly one point. (except when parallel)*

Theorem 8. *A line meets a parabola in exactly two points. (except when false)*

We need projective space to remove the bad cases.

Definition 18 (Projective Space)**.** *Define an equivalence relation on* $\mathbb{A}^{n+1} \setminus \{0\}$ *by*

$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ *where* $\lambda \in k^* = k \setminus \{0\}$.

So $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\sim$ *and* $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ *the projection.*

There is a topology on $\mathbb{P}^n$ given by $U \subseteq \mathbb{P}^n$ is open iff $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ is open.

The regular functions on $\mathbb{P}^n$ are $f : U \to k$ such that $\pi^*(f) = \pi \circ f : \pi^{-1}(U) \to k$ is regular.

Thus, $\mathbb{P}^n$ is a SWF called Projective Space. Note: $\mathbb{P}^n = \{$lines through the origin in $\mathbb{A}^{n+1}\}$, and this method of thinking is often very helpful.

We will use the notation $(a_0 : \ldots : a_n)$ for the image $\pi(a_0, \ldots, a_n) \in \mathbb{P}^n$. If $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of total degree d, then $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$. Thus it is well-defined to ask if $f(a_0 : \ldots : a_n) = 0$ or not.

Definition 19. $D_+(f) = \{(a_0 : \ldots : a_n) \in \mathbb{P}^n : f(a_0 : \ldots : a_n) \neq 0\}$.

Theorem 9. $\mathbb{P}^n$ *is a prevariety.*

Proof. Let $U_i = D_+(x_i) \subseteq \mathbb{P}^n$ for $0 \leq i \leq n$.

Claim: $U_i \simeq \mathbb{A}^n$.

$\phi : \mathbb{A}^n \to U_i : (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \mapsto (a_0 : \ldots : a_{i-1} : 1 : a_{i+1} : \ldots : a_n)$

$$\psi : U_i \to \mathbb{A}^n : (a_0 : \ldots : a_n) \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{\hat{a}_i}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

□

Note:$\mathbb{P}^n = D_+(x_0) \coprod V_+(x_0) = \mathbb{A}^n \coprod \mathbb{P}^{n-1}$, that is, $\mathbb{A}^n = \{(1 : a_1 : \ldots a_n)\}$ the usual n-space and $\mathbb{P}^{n-1} = \{0 : a_1 : \ldots : a_n\}$ points at ∞.

The points at ∞ correspond to lines through the origin in $\mathbb{A}^n$, that is, we can think of them as being directions.

Example: $\mathbb{P}^2$ has "homogeneous coordinate ring" $k[x, y, z]$. We can think of that as $\mathbb{A}^2 = D_+(z) \subset \mathbb{P}^2$ and we know that $k[\mathbb{A}^2] = k[x/z, y/z]$. We want to intersect a parabola with a line.

The vertical line is $\overline{V(x/z - 1)} = V_+(x-z)$ and the parabola is $\overline{V(y/z - (x/z)^2)} = V_+(yz - x^2)$.

And so, $V_+(x-z) \cap V_+(yz-x^2) = \{(1 : 1 : 1), (0 : 1 : 0)\}$, where $(0 : 1 : 0)$ is the point at infinity in the direction "up".

Exercise: $k[\mathbb{P}^n] = k$.

Exercise: X a SWF, $\phi : \mathbb{P}^n \to X$ a function, then ϕ is a morphism iff $\phi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X$ is a morphism.

Definition 20 (Projective Coordinate Ring of $\mathbb{P}^n$)**.** *We define $k[x_0, x_1, \ldots, x_n]$ to be the coordinate ring of $\mathbb{P}^n$. An ideal $I \subseteq k[x_0, \ldots, x_n]$ is homogeneous if it is generated by homogeneous polynomials. Equivalently, $f \in I$ iff each homogeneous component is in I.*

Definition 21. *If $W \subseteq \mathbb{P}^n$ is a subset, then $I(W) = I(\pi^{-1}(W)) \subseteq k[x_0, \ldots, x_n]$. Notice that $I(W)$ is homogeneous. Let $f = f_0 + \ldots + f_d \in I(W)$, f_i a form of degree i, then $(a_0 : \ldots : a_n) \in W$, so $0 = f(\lambda a_0, \ldots, \lambda a_n) = f_0(a_0, \ldots, a_n) + \ldots + \lambda^d f_d(a_0, \ldots, a_n)$. As this is true for all λ, $f_i(a_0, \ldots, a_n) = 0$ for all i, and so $f_i \in I(W)$.*

Definition 22. *If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal, then define $V_+(I) = \{(a_0 : \ldots, a_n) \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0$ for all $f \in I\}$*

Theorem 10 (Projective Nullstellensatz). *If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal, then*

1. *$V_+(I) = \emptyset \Rightarrow (x_0, \ldots, x_n)^N \subseteq I$ for some $N > 0$. That is, $\sqrt{I} = (1)$ or $(x_0, \ldots, x_n)$.*
2. *$V_+(I) \neq \emptyset$ then $I(V_+(I)) = \sqrt{I}$.*

Proof.

1. $V_+(I) = \emptyset \iff V(I) = \emptyset$ or $V(I) = \{0\}$. By the regular nullstellensatz, $\sqrt{I} = I(V(I)) = (1)$ or $(x_0, \ldots, x_n)$.
2. $V_+(I) \neq \emptyset$. Then $\overline{\pi^{-1}(V_+(I))} = \pi^{-1}(V_+(I)) \cup \{0\} = V(I) \subseteq \mathbb{A}^{n+1}$. So $I(V_+(I)) = I(V(I)) = \sqrt{I}$.

□

This gives us a 1-1 correspondence between closed subsets of $\mathbb{P}^n$ and radical homogeneous ideals in $k[x_0, \ldots, x_n]$ except for $(x_0, \ldots, x_n)$. This ideal is often called the irrelevant ideal.

Definition 23 (Locally Closed subset). *X is a topological space, $W \subseteq X$ a subset is locally closed if it is the intersection of an open set in X and a closed set in X.*

Note: A locally closed subset of a prevariety is a prevariety.

Terminology: a projective variety is any closed subset of $\mathbb{P}^n$ considered as a space with functions. A Quasi-projective variety is a locally closed subset of $\mathbb{P}^n$. An affine variety is a closed subset of $\mathbb{A}^n$. A quasi-affine variety is a locally closed subset of $\mathbb{A}^n$.

We notice that anything affine is also quasi-affine and anything quasi-affine is quasi-projective. Something that is projective will also be quasi-projective.

Exercise: $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$. Later: If X is both projective and quasi-affine, then X is finite.

Definition 24 (Projective Coordinate Ring). *$X \subseteq \mathbb{P}^n$ is a closed projective variety, then the projective coordinate ring of $X = k[x_0, \ldots, x_n]/I(X)$.*

Warning: This definition depends on the embedding of X in $\mathbb{P}^n$.

Example: $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ by $\phi(a : b) = (a^2 : ab : b^2)$. This is a morphism. In fact, it is an isomorphism of $\mathbb{P}^1$ and $V_+(xz - y^2)$, but the coordinate ring of $\mathbb{P}^1$ is $k[s, t]$ and the coordinate ring of $V_+(xz - y^2)$ is $k[x, y, z]/(xz - y^2)$. These two rings are NOT isomorphic as k-algebras.

Definition 25 (Projective Closure of an affine variety)**.** *$X \subseteq \mathbb{A}^n$ is affine. Then we know that $\mathbb{A}^n = D_+(x_0) \subsetneq \mathbb{P}^n$, and $X \subset \mathbb{A}^n \subseteq \mathbb{P}^n$ makes X a quasi-projective variety, so we take $\bar{X}$ = the closure of X in $\mathbb{P}^n$.*

$I = I(X) \subseteq k[x_1, \ldots, x_n]$ if $f = f_0 + \ldots + f_d \in k[x_1, \ldots, x_n]$ where f_i is a form of degree i. Then we define $f^* = x_0^d f_0 + x_0^{d-1} f_1 + \ldots + f_d \in k[x_0, \ldots, x_n]$. And I^* is the ideal generated by $\{f^* : f \in I\}$ in $k[x_0, \ldots, x_n]$.

Exercise: $I(\bar{X}) = I(X)^*$.

Example: $I = (y - x^2, z - x^2) \subseteq k[x, y, z]$. Then $X = V(I) \subseteq \mathbb{A}^3 = D_+(w) \subset \mathbb{P}^n$. $I(\overline{X}) = I^* = (yw - x^2, y - z) \supsetneq (wy - x^2, wx - x^2)$

So $V_+(wy - x^2, wz - x^2) = \bar{X} \cup V_+(x, w)$.

We now recall that a graded ring is a ring R with decomposition $R = \oplus_{d \geq 0} R_d$ as an abelian group such that $R_d \cdot R_e \subseteq R_{d+e}$.

e.g., $R = k[x_0, \ldots, x_n]$. $f \in R_d \Rightarrow R_f$ is a $\mathbb{Z}$-graded ring $g \in R_p$ implies that $g/f^m \in R_f$ is homogeneous of degree $p - md$.

Definition 26. *$R_{(f)} = \{$homogeneous elements of degree zero$\} = (R_f)_0 = \{g/f^m : g \in R_{dm}\}$.*

Exercise: $f \in k[x_0, \ldots, x_n]$ homogeneous implies that $k[x_0, \ldots, x_n]_{(f)}$ is a finitely generated reduced k-algebra.

Theorem 11. *$f \notin k \Rightarrow D_+(f) \subseteq \mathbb{P}^n$ is affine and in fact $k[D_+(f)] = k[x_0, \ldots, x_n]_{(f)}$.*

Proof. $k[D_+(f)] = \{h \in k[D(f)] : h(\lambda x) = h(x), \forall \lambda \in k^*, x \in D(f)\}$.

If $h \in k[D(f)] = k[x_0, \ldots, x_n]_f$, $h = g/f^m$, $g \in k[x_0, \ldots, x_n]$.

$$\frac{g}{f^m}(\lambda a_0, \ldots, \lambda a_n) = \frac{g}{f^m}(a_0, \ldots, a_n) \iff g \texttt{ homogeneous of degree } md$$

Therefore, $k[D_+(f)] = k[x_0, \ldots, x_n]_{(f)}$.

The identity map $k[D_+(f)] \to k[D_+(f)]$ gives a morphism $\phi : D_+(f) \to \operatorname{Spec} -m(k[D_+(f)])$ by $\phi(x) = M_x$ where $M_x = I(\{x\}) \subseteq k[D_+(f)]$

Observe that if $x, y \in D_+(f)$, $x \neq y$ then $\exists$ homogeneous $g \in k[x_0, \ldots, x_n]$ such that $\deg(g) = d$ and $g(x) = 0$ with $g(y) \neq 0$. So $M_x \neq M_y$. $\frac{g}{f} \in M_x, \notin M_y$. Thus, ϕ is injective. Set $h_i = \frac{x_i^d}{f} \in k[D_+(f)]$ for $0 \leq i \leq n$.

$U_i = D(h_i) \subseteq D_+(f)$, $V_i = D(h_i) \subseteq \operatorname{Spec} -m(k[D_+(f)]) = \{m \not\ni h_i\}$.

Now we must check that $D_+(f) = \cup_{i=0}^n U_i$ and $\operatorname{Spec}-m(k[D_+(f)]) = \cup_{i=0}^n V_i$.

It is enough to prove that $\phi : U_i \to V_i$ is an isomorphism for all i.

$D_+(x_i) \subseteq \mathbb{P}^n$ is affine. $k[D_+(x_i)] = k[x_0/x_i, \ldots, x_n/x_i] = k[x_0, \ldots, x_n]_{(x_i)}$.

Thus, $U_i = D(f/x_i^d) \subseteq D_+(x_i)$ is affine, so $k[U_i] = (k[x_0, \ldots, x_n]_{(x_i)})_{f/x_i^d}$, which is $k[x_0, \ldots, x_n]_{(x,f)} = k[D_+(f)]_{h_i} = k[V_i]$. Thus, $U_i \simeq V_i$. $\square$

Example: $f = xz - y^2 \in k[x, y, z]$. $X = D_+(f) \subseteq \mathbb{P}^2$, $R = k[x, y, z]_{(f)}$. R is generated by $A = x^2/f$, $B = y^2/f$, $C = z^2/f$, $D = xy/f$, $E = yz/f$, and $F = xz/f$.

So $X \simeq V(AB - D^2, AC - F^2, BC - E^2, F - B - 1) \subseteq \mathbb{A}^6$ by $(x : y : z) \mapsto (A, B, C, D, E, F)$.

Exercise:$X \subseteq \mathbb{P}^n$ a projective variety $f \in R = k[x_0, \ldots, x_n]/I(X)$ is homogeneous, then $D_+(f) \subseteq X$ is affine with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Chapter 3

Abstract Varieties

3.1 Products

Let X, Y be two sets. Then $X \times Y$, the cartesian product, is the set $\{(x, y) : x \in X, y \in Y\}$.

What is $X \times Y$, really? Well, it is a set with projection $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. This set with the projections satisfies a universal property in the category of sets.

For any set Z with arbitrary functions $f : Z \to X$ and $g : Z \to y$, there exists a unique function $\phi : Z \to X \times Y$ such that $f = \pi_X \circ \phi$ and $g = \pi_Y \circ \phi$.

$$\begin{tikzcd} & & X \\ Z \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, dashed, "\exists!"] & X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ & & Y \end{tikzcd}$$

Definition 27 (Product of SWFs)**.** *Let X, Y be spaces with functions. A product of X and Y is a SWF called $X \times Y$ with morphism $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ which satisfies the above universal property except with "morphisms" rather than "functions".*

<u>Exercise</u>: Assume that (P, π_X, π_Y) and (P', π'_X, π'_Y) are two products of X and Y. Then they are isomorphic by unique isomorphism. (See homework problem)

<u>Example</u>: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. NOTE: $\mathbb{A}^2$ does not have the product topology!

3.1.1 General Construction

X, Y spaces with functions. Then $X \times Y = \{(x,y) : x \in X, y \in Y\}$ is the point set. If $U \subset X$ and $V \subset Y$ are open, then $U \times V \subset X \times Y$ is open, as it is $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$.

Let $g_1, \ldots, g_u \in \mathscr{O}_X(U)$ and $h_1, \ldots, h_n \in \mathscr{O}_Y(V)$ set $f(u,v) = \sum_{i=1}^n g_i(u)h_i(v)$. Then $f : U \times V \to k$ must be regular. Thus, $D_{U\times V}(f) = \{(u,v) \in U \times V : f(u,v) \neq 0\}$ must be open in $X \times Y$. And so, we define our topology by $S \subseteq X \times Y$ is open iff it is a union of sets $D_{U\times V}(f)$. The regular functions $F : S \to k$ are the functions that can locally be written as $f'(u,v)/f(u,v)$ on some $D_{U\times V}(f)$.

That is, $\forall (x,y) \in S$ $\exists U \subseteq X$ open and $V \subseteq Y$ open and $f(u,v) = \sum_{i=1}^n g_i(u)h_i(v)$ and $f'(u,v) = \sum_{j=1}^m g'_j(u)h'_j(u)$ with $g_i, g'_j \in \mathscr{O}_X(U)$ and $h_i, h'_j \in \mathscr{O}_Y(V)$ such that $(x,y) \in D_{U\times V}(f) \subseteq S$ and $F(u,v) = f'(u,v)/f(u,v)$ for all $(u,v) \in D_{U\times V}(f)$.

<u>Exercises</u> (for $X \times Y$ above)

1. $X \times Y$ is an SWF
2. $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are morphisms.
3. $X \times Y$ is the product of X and Y.

Remark: X, Y SWFs, and $U \subseteq X$, $V \subseteq Y$ are arbitrary subsets, U, V have inherited structure as SWFs. Then $U \times V$ has the product space with functions structure and subspace SWF structure $U \times V \subseteq X \times Y$. These are in fact the same, due to the universal properties.

For now, we call $U \times V$ the product. We obtain the following diagram.

$$\begin{tikzcd}
 & U \arrow[rr] & & X \\
Z \arrow[r, "\phi"] & U \times V \arrow[u, "\pi_U"] \arrow[d, "\pi_V"'] \arrow[rr, dashed, "i"] & & X \times Y \arrow[u, "\pi_X"] \arrow[d, "\pi_Y"] \\
 & V \arrow[rr] & & Y
\end{tikzcd}$$

ϕ is a morphism iff $i \circ \phi$ is one, and so we see that the two structures are the same.

3.2 Separated spaces with functions

Definition 28. *A SWF X is separated if $\forall$ SWFs Y and morphisms $f, g : Y \to X$ the set $\{y \in Y : f(y) = g(y)\} \subseteq Y$ is closed.*

Example: Let $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{O_1, O_2\}$. We can define $\phi_i : \mathbb{A}^1 \to X$ by taking $a \mapsto \begin{cases} a & a \neq 0 \\ O_i & a = 0 \end{cases}$

We define a topology by $U \subseteq X$ is open iff $\phi_i^{-1}(U) \subseteq \mathbb{A}^1$ is open for all i. A function $f : U \to k$ is regular iff $\phi_i^*(f) = f \circ \phi_i : \phi_i^{-1}(U) \to k$ is regular for all i.

X is a prevariety as $X = \phi_1(\mathbb{A}^1) \cup \phi_2(\mathbb{A}^1)$ and $\phi_i(\mathbb{A}^1) \simeq \mathbb{A}^1$. However, it is not separated, as $\{a \in \mathbb{A}^1 : \phi_1(a) = \phi_2(a)\} = \mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$.

Definition 29 (Algebraic Variety)**.** *An algebraic variety is a separated prevariety.*

Exercise:

1. Any subspace of a separated SWF is separated.
2. A product of separated SWFs is separated.

Remark: If X is any SWF, then $\Delta : X \to X \times X : x \mapsto (x, x)$ is a morphism. Now we set $\Delta_X = \Delta(X) \subseteq X \times X$. Then $\Delta : X \to \Delta_X$ is an isomorphism.

Lemma 2. *X is separated iff $\Delta_X \subseteq X \times X$ is closed.*

Proof. $\Rightarrow$: $\pi_i : X \times X \to X$ be the projections. $\Delta_X = \{z \in X \times X : \pi_1(z) = \pi_2(z)\}$ is closed.

$\Leftarrow$: Let Y be a SWF, $f, g : Y \to X$ maps. Define $\phi : Y \to X \times X$ by $\phi(y) = (f(y), g(y))$ is a morphism. Now $\{y \in Y : f(y) = g(y)\} \Rightarrow \phi^{-1}(\Delta_X)$ is closed. $\square$

Exercise: A topological space X is Hausdorff iff $\Delta_X \subseteq X \times X$ is closed.
NB: Product topology!
Exercise: $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.

Proposition 8. *All affine varieties are varieties.*

Proof. Enough to show that $\mathbb{A}^n$ itself is separated.

$\Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ is closed, as $k[\mathbb{A}^{2n}] = [x_1, \ldots, x_n, y_1, \ldots, y_n]$, so $\Delta_{\mathbb{A}^n} = V(\{x_i - y_i\})$. $\square$

3.3 Affine and projective products

Lemma 3. *Let A, B be finitely generated reduced k-algebras. Then $A \otimes_k B$ is a finitely generated reducted k-algebra and k algebraically closed.*

Recall: ring structure on $A \otimes B$ by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Proof. If A is generated by $a_1, \dots, a_n$ and B is generated by $b_1, \dots, b_m$ then $A \otimes B$ is generated by $a_1 \otimes 1, \dots, a_n \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_m$. For example, $k[x_1, \dots, x_n] \otimes k[y_1, \dots, y_m] = k[x_1, \dots, x_n, y_1, \dots, y_m]$.

Let X, Y be affine varieties such that $k[X] = A, k[Y] = B$. We define $\phi : A \otimes_k B \to$the set of all functions $X \times Y \to k$ by $f \otimes g \mapsto \big[(x, y) \mapsto f(x)g(x)\big]$ is a k-algebra homomorphism.

ϕ is injective (which implies that $A \otimes B$ is reduced): Suppose $\phi\left(\sum_{i=1}^n f_i \otimes g_i\right) = 0$. WLOG we can assume $g_1, \dots, g_n$ are linearly independent. Let $x \in X$. Then $\sum_{i=1}^n f_i(x) g_i = 0 \in B$, but $\{g_i\}$ is linearly independent so $f_i(x) = 0$ for all i and x, thus $f_i = 0 \in A$. $\square$

Theorem 12. *1. If X, Y are affine then $X \times Y$ is affine and $k[X \times Y] = k[X] \otimes_k k[Y]$.*

2. A product of prevarieties is a prevariety.

Proof. $1 \Rightarrow 2$: X, Y prevarieties, $X = \cup U_i$, $Y = \cup V_j$ with U_i, V_j affine. Then $X \times Y = \cup_{i,j} U_i \times V_j$ affine.

Now, we must only prove 1. Set $P = \text{Spec-}m(k[X] \otimes k[Y])$, We have k-algebra homomorphisms $k[X] \to k[X] \otimes k[Y] : f \mapsto f \otimes 1$ and $k[Y] \to k[X] \otimes k[Y] : g \mapsto 1 \otimes g$. These give morphism $\pi_X : P \to X$ and $\pi_Y : P \to Y$. Claim: $P = X \times Y$. Let Z be a SWF, $p : Z \to X$ and $q : Z \to Y$.

Define k-alg homomorphism $k[X] \mapsto k[Y] \to k[Z]$ by $f \otimes g \mapsto p^*(f) q^*(g)$. This gives us a morphism $\phi : Z \to P$ demonstrating that P is the product. $\square$

Remark: If Y is affine and $X \subseteq Y$ is closed, then $k[X] \simeq k[Y]/I(X)$. On the other hand, assume X, Y are affine, $\varphi : X \to Y$ is a morphism and $\varphi^* : k[Y] \to k[X]$ is surjective. Then set $I = \ker(\varphi^*) \subseteq k[Y]$, we get

$$\begin{tikzcd} k[Y] \arrow[rr, "\varphi^*"] \arrow[dr] & & k[X] \\ & k[Y]/I \arrow[ur, "\simeq"'] & \end{tikzcd} \qquad \begin{tikzcd} Y & & X \arrow[ll, "\varphi"'] \arrow[dl, "\simeq"] \\ & V(I) \arrow[ul, "\text{inj}"] & \end{tikzcd}$$

Therefore φ is an embedding of X as a closed subset of Y.

Recall: $\Delta : X \to X \times X : x \mapsto (x, x)$ gives $X \simeq \Delta_X = \Delta(X) = \{(x, x) : x \in X\}$.

Proposition 9. *A prevariety X is separated iff $\forall$ open affine $U, V \subseteq X$, $U \cap V$ is affine and $k[U \times V] = k[U] \otimes k[V] \to k[U \cap V] = k[\Delta_{U \cap V}]$ is surjective.*

Proof. $\Rightarrow$: $U \cap V \simeq \Delta_{U \cap V} = \Delta_X \cap (U \times V) \subseteq U \times V$ is closed, thus $U \cap V$ is affine and $k[U \times V] \to k[U \cap V]$ is surjective.

$\Leftarrow$: If $U, V, U \cap V$ are affine and $k[U \times V] \to k[U \cap V]$ is surjective, then $\Delta : U \cap V \to U \times V$ is an inclusion of closed subsets. Thus, $\Delta_X \cap (U \times V) \subseteq U \times V$ closed. So if $X \times X = \cup U \times V$ is an open cover then $\Delta_X \subseteq X \times X$ is closed. $\square$

Exercises

1. X is a prevariety such that $\forall x, y \in X$ there is an open affine $U \subseteq X$ such that $x, y \in U$. Then X is separated.

2. $\mathbb{P}^n$ has this property.

Corollary 6. *Quasi-projective varieties are separated.*

We want to show that the products of projective varieties are again projective.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are closed. Then $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed. It is enough to show that $\mathbb{P}^n \times \mathbb{P}^m$ is projective, that is, $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$ is closed.

Segre Map: If $N = (n+1)(m+1) - 1 = nm + n + m$, then we can define $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N : (x_0 : \ldots : x_n) \times (y_0 : \ldots : y_m) \mapsto (x_0 y_0 : x_0 y_1 : \ldots : x_0 y_m : x_1 y_0 : \ldots : x_n y_m)$.

We call the projective coordinates on $\mathbb{P}^N$ as z_{ij} for $0 \leq i \leq n, 0 \leq j \leq m$.

Exercise: $s : \mathbb{P}^n \times \mathbb{P}^m \to V_+(\{z_{ij} z_{pq} - z_{iq} z_{pj}\}) \subseteq \mathbb{P}^N$

Note: $\mathbb{P}^n \times \mathbb{P}^n \subseteq \mathbb{P}^{n^2+2n}$ is closed.

Exercise: $\Delta_{\mathbb{P}^n} = V_+(\{z_{ij} - z_{ji}\}) \subseteq \mathbb{P}^N$.

3.4 Complete Varieties

Aanalogues of compact manifolds.

Definition 30 (Complete)**.** *A variety X is complete if for any variety Y, the projection $\pi_Y : X \times Y \to Y$ is closed. (i.e.: $Z \subseteq X \times Y$ is closed implies that $\pi_Y(Z) \subseteq Y$ is closed)*

Note: 1) closed subsets of complete varieties are complete.

2) Products of complete varieties are complete.

Examples:Points are complete.

$\mathbb{A}^1$ is not complete, as $Z = V(xy-1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 = X \times Y$ is such that $\pi_Y(Z)$ is not closed, as it is $\mathbb{A}^1 \setminus \{0\}$

Proposition 10. *Let $\varphi : X \to Y$ be a morphism of varieties. If X is complete, then $\varphi(X)$ is closed in Y and is complete.*

Proof. $\Gamma(\varphi) = \{(x, \varphi(x)) \in X \times Y : x \in X\} \subseteq X \times Y = (\varphi \times 1)^{-1}(\Delta Y)$

As Y is separated, $\Gamma(\varphi) \subseteq X \times Y$ is closed. X is complete implies that $\varphi(X) = \pi_Y(\Gamma(\varphi)) \subseteq Y$ is closed.

Now, let $Z \subseteq \varphi(X) \times Y'$ be closed. Then

$$\begin{tikzcd} X \times Y' \arrow[r, "\varphi \times 1"] \arrow[dr, "\tilde{\pi}_{Y'}"'] & \varphi(X) \times Y' \arrow[d, "\pi_{Y'}"] \\ & Y' \end{tikzcd}$$

$W = (\varphi \times 1)^{-1}(Z) \subseteq X \times Y'$ is closed, $\pi_{Y'}(Z) = \pi_{Y'}((\varphi \times 1)(W)) = \tilde{\pi}_{Y'}(W)$ is closed. □

Exercise: $\varphi : X \to Y$ is a continuous map of topological spaces then X is irreducible implies that $\varphi(X)$ is irreducible.

Proposition 11. *If X is an irreducible complete variety then $k[X] = k$.*

Proof. Let $f \in k[X]$. $f : X \to \mathbb{A}^1$ a morphism. As X is complete, $f(X)$ must be is irreducible, closed and complete. Thus, it must be a point. Thus, f is constant. □

Proposition 12. *A complete quasi-affine variety is finite.*

Proof. X is such a variety, without loss of generality X is irreducible. $X \subseteq \mathbb{A}^n$ is locally closed, then $x_i : X \subseteq \mathbb{A}^n \to k$ must be constant, and so x is a point. $\square$

Theorem 13. *$\mathbb{P}^n$ is complete.*

Note: $I \subseteq S = k[x_0, \dots, x_n]$ a homogeneous ideal, then $V_+(I)$ is not empty iff $I_d \subsetneq S_d$ for all $d \in \mathbb{N}$.

Proof. Let Y be a variety and $Z \subseteq \mathbb{P}^n \times Y$ be closed. Show that $\pi_Y(Z) \subseteq Y$ is closed.

$Y = \cup Y_i$ and open affine cover. It is enough to show that $\pi_Y(Z) \cap Y_i \subseteq Y_i$ is closed, that is, $\pi_{Y_i}(Z \cap (\mathbb{P}^n \times Y_i))$ is closed. So we take $\pi_{Y_i} : \mathbb{P}^n \times Y_i \to Y_i$.

Thus, WLOG, we assume Y is affine. Then let $C(Z) = (\pi \times \mathrm{id})^{-1}(Z) \subseteq \mathbb{A}^{n+1} \times Y$. $k[\mathbb{A}^{n+1} \times Y] = S \otimes k[Y] = k[Y][x_0, \dots, x_n] = \oplus_{d \geq 0} S_d \otimes_k k[Y]$, so it is a graded ring.

Note that $(y, (a_0, \dots, a_n)) \in C(Z)$ implies that $(y, (\lambda a_0, \dots, \lambda a_n)) \in C(Z)$ for $\lambda \in k$.

Thus, $I(C(Z)) \subseteq k[\mathbb{A}^{n+1} \times Y]$ is a homogeneous ideal. We write $I(C(Z)) = (f_1, \dots, f_m)$ with $f_i \in S_{d_i} \otimes k[Y]$.

For $y \in Y$, $f_i(y) = f_i(-, y) \in S_{d_i}$. We observe that $y \in \pi_Y(Z)$ iff $\exists x \in \mathbb{P}^n$ such that $(x, y) \in Z$. This happens iff $V_+(f_1(y), \dots, f_m(y)) \neq \emptyset \subseteq \mathbb{P}^n$. This is true iff $(f_1(y), \dots, f_m(y))_d \neq S_d$ for all $d \geq 0$.

Fix $d \geq 0$, then $y \in Y$ defines a linear map $\Phi_Y : \oplus_{i=1}^m S_{d-d_i} \to S_d : (g_1, \dots, g_m) \mapsto \sum_{i=1}^m f_i(y) g_i$.

Note: Every entry of the matrix Φ_Y is a regular function of Y.

Now, $(f_1(y), \dots, f_m(y))_d \neq S_d$ iff $\mathrm{rank}(\Phi_Y) < \dim(S_d) = \binom{n+d}{n}$ which holds iff all minors in Φ_Y of size $\binom{n+d}{n}$ vanish. Therefore, $W_d = \{y \in Y : (d_1(y), \dots, f_m(y))_d \neq S_d\} \subseteq Y$ is closed. Finally, $\pi_Y(Z) = \cap_{d \geq 0} W_d$, and so is closed. $\square$

Challenge: Find a complete variety that is not projective.

Exercise:

1. Let X be a topological space and $W \subseteq X$ is a subset. $\overline{W} = X$ iff $W \cap U \neq \emptyset$ for all nonempty open sets $U \subseteq X$.

2. $f : X \to Y$ continuous and $\overline{W} = X$ and $\overline{f(X)} = Y$ then $\overline{f(W)} = Y$

3. X is irreducible and $\emptyset \neq U \subseteq X$ is open. Then $\overline{U} = X$ and U is irreducible.

3.5 Rational Map

X and Y are irreducible varieties. The ideal is that a morphism $f : X \to y$ is uniquely determined by restriction to any non-empty open subset of X.

Consider pairs (U, f) where $U \subseteq X$ nonempty and open and $f : U \to Y$ is a morphism. Relation: $(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$ because Y is separated and X is irreducible, this is an equivalence relation. Checking this is an exercise.

Definition 31 (Rational Map)**.** *A rational map $f : X \dashrightarrow Y$ is an equivalence class for $\sim$.*

X irreducible implies that $U \cap W \supseteq U \cap V \cap W$ dense, and Y separated implies that $f = h$ on a closed subset of $U \cap W$.

Remark: If $f : X \dashrightarrow Y$ then there is a maximal open $U \subseteq X$ where f is defined as a morphism. $U = \cup_{(V,g)\sim f} V \subset X$.

Example: $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2 : (x, y) \mapsto (x/y, y/x^2)$ defined as a morphism of $D(xy)$.

Exercise: $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2 \subseteq \mathbb{P}^2$. Find the max open where $f : \mathbb{A}^2 \dashrightarrow \mathbb{P}^2$ is defined.

Definition 32 (Rational Function)**.** *A rational function on X is a rational map $f : X \dashrightarrow \mathbb{A}^1 = k$. f is given by a regular function $f : U \to k$ where $\emptyset \neq U \subseteq X$ open.*

$k(X) = \{f : X \dashrightarrow k\}$ is the field of rational functions on X.

Note: If $(U, f), (V, g) \in k(X)$ then $f + g, f - g, fg : U \cap V \to k$ define rational functions on X. If $f \neq 0$ in $k[U]$ then $\emptyset \neq D(f) \subseteq U$ is open, and $1/f : D(f) \to k$ is regular. Thus, $1/f = (D(f), 1/f) \in k(X)$.

Examples: $k(\mathbb{A}^n) = k(x_1, \ldots, x_n)$. $k(\mathbb{P}^n) = k(x_1/x_0, \ldots, x_n/x_0)$.

Proposition 13. *Let X be an irreducible variety*

1. *If $\emptyset \neq U \subseteq X$ open, then $k(X) = k(U)$.*

2. *If X is affine, then $k(X) = k[X]_0$ =field of fractions of $k[X]$.*

Proof. 1. $k(X) \to k(U) : (V, g) \to (V \cap U, g|_{V\cap U})$ is isomorphism.

2. Define $k[X]_0 \to k(X) : f/g \mapsto (D(g), f/g)$.

   Injective: As this is a homomorphism of fields, it is enough to say that it is not identically zero, and it maps 1 to $(X, 1)$, which is not the zero function.

   Surjective: If $f : U \to k$ is regular, $\emptyset \neq U \subseteq X$ open. Find $0 \neq g \in k[X]$ such that $\emptyset \neq D(g) \subseteq U$. Then $f \in k[D(g)] = k[X]_g \subseteq k[X]_{x_0}$.

□

Definition 33 (Dominant)**.** $(U, f) : X \dashrightarrow Y$ *is dominant if* $\overline{f(U)} = Y$*.*

Indep. of rep.: If $\emptyset \neq V \subseteq U$ open, then $\overline{f(V)} = Y$ by the homework.

Suppose $(U, f) : X \dashrightarrow Y$ is dominant and $(V, g) : Y \dashrightarrow Z$ is any rational map, then we can compose $g \circ f : X \dashrightarrow Z$, as $\overline{f(U)} = Y$ so $f(U) \cap V \neq \emptyset$, so $f^{-1}(V) \neq \emptyset \subseteq U$.

$g \circ f = (f^{-1}(V), g \circ f)$.

Exercise: If f, g both dominant, then $g \circ f$ dominant.

Proposition 14. *X, Y irreducible varieties, then there is a one to one correspondence between $\{\phi : X \to Y$ with ϕ dominant$\}$ and field extensions $k(Y) \subseteq k(X)$ by $\phi \mapsto \phi^* = [h \mapsto h \circ \phi]$*

Proof. WLOG X, Y are affine. The map $\phi \mapsto \phi^*$ is:

Injective: So we let $\psi : X \dashrightarrow Y$ dominant and $\psi^* = \phi^*$. Take $D(h) \subseteq X$ such that ϕ and ψ are both defined on $D(h)$.

$$\begin{array}{ccc} k(Y) & \xrightarrow{\phi^* = \psi^*} & k(X) \\ \subseteq\uparrow & & \subseteq\uparrow \\ k[Y] & \longrightarrow & k[X]_h \end{array}$$

This diagram commutes, and so $\phi = \psi$ on $D(h)$.

Surjective: Let $\alpha : k(Y) \to k(X)$ be a k-algebra homomorphism. $k[Y]$ is generated by $f_1, \dots, f_n$. $\alpha(f_i) = g_i/h_i$, $g_i, h_i \in k[X]$. Let $h = h_1 \dots h_n \in k[X]$. So $\alpha : k[Y] \to k[X]_h$ is a k-alg hom. This gives us a morphism $\phi : D(h) \to Y$ and $\phi^* = \alpha$. □

Definition 34 (Birational)**.** *Let $f : X \dashrightarrow Y$ be a rational map. It is birational if f is dominant and there is a dominant $g : Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ as rational maps.*

Definition 35 (Birationally Equivalent). *X and Y are birationally equivalent (often X and Y are birational) written $X \approx Y$ if there exists $f : X \dashrightarrow Y$ a birational map.*

Examples: $\mathbb{A}^2 \approx \mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$.
If $U, V \subseteq X$ open and X irred., then $U \approx Y$.

Theorem 14. *The following are equivalent*

1. $X \approx Y$
2. $k(X) \simeq k(Y)$ *as k-algebras*
3. $\exists \emptyset \neq U \subset X, V \subset Y$ *open such that $U \simeq V$ as varieties.*

Proof. $3 \Rightarrow 2$ is clear from the first prop on this topic.

$2 \Rightarrow 1$ is clear from the second prop on this topic.

$1 \Rightarrow 3$: Let $(U, f) : X \dashrightarrow Y$ and $(V, g) : Y \dashrightarrow X$ be inverses. Set $U_0 = f^{-1}(V) \subseteq U$. Then $g \circ f : U_0 \to V \to X$ must be the inclusion of U_0 into X. Thus, $g(f(U_0)) \subseteq U_0$, so $f(U_0) \subseteq g^{-1}(U_0)$. Set $V_0 = g^{-1}(U_0) \subseteq V$. Then $U_0 \simeq V_0$ as varieties by f, g. $\square$

Definition 36 (Rational Variety). *An irreducible variety is rational if it is birational to $\mathbb{A}^n$ for some n.*

Examples: Any curve $C \subseteq \mathbb{P}^2$ of degree 2 is rational.

Let $C = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$ is rational. Let $\phi : C \dashrightarrow \mathbb{A}^1$ by $\phi(x, y) = y/x$. The inverse should be $\psi : \mathbb{A}^1 \to C$ by $\psi(t) = (1 - t^2, t - t^3)$. So $\phi \circ \psi(t) = (t - t^3)/(1 - t^2) = t = \mathrm{id}_{\mathbb{A}^1}$, and the opposite is also an identity (Exercise, show this).

Challenge: Show that $E = V(y^2 - x^3 + x) \subseteq \mathbb{A}^2$ is not rational.

Big Challenge: If C is any irreducible variety and there exists a dominant rational map $\mathbb{A}^1 \dashrightarrow C$ then C is rational.

Chapter 4

Dimension

4.1 Transcendence Degree

Let $k \subseteq L$ a field extension. Then L is algebraic over k if for all $f \in L$, there is a polynomial equation $f^n + a_1 f^{n-1} + \ldots + a_n = 0$ for $a_i \in k$.

$S \subseteq L$ subseteq, then S is algebraically independent over k if for all $s_1, \ldots, s_n \in S$ with $s_i \neq s_j$ for $i \neq j$, then $k[x_1, \ldots, x_n] \to L$ by $x_i \mapsto s_i$ is injective.

Definition 37 (Transcendence Basis)**.** *A transcendence basis for L over k is a set $B \subseteq L$ such that B is algebraically independent over k and L is algebraic over $k(B)$.*

Theorem 15.

1. *All transcendence bases have the same cardinality.*
2. *If $S \subseteq \Gamma \subseteq L$ subsets such that S is alg indep over k and L is alg over $k(\Gamma)$, then there exists a transcendence basis B for L over k such that $S \subseteq B \subseteq \Gamma$.*

Proof. The idea is as "any vector space has a basis." □

<u>Exercise</u>: Prove where L is a finitely generated extension of k.
Lang's Algebra contains the proof.

Definition 38 (Transcendence Degree)**.** *The transcendence degree* $\operatorname{tr}\deg_k(L) = \operatorname{tr}\deg(L) =$ *the number of elements in any transcendence basis for L over k.*

Definition 39 (Dimension)**.** *Let X be an irreducible variety. Then define* $\dim(X) = \operatorname{tr}\deg(k(X))$.

Examples:

1. $\dim(\mathbb{A}^n) = \operatorname{tr}\deg_k k(x_1, \ldots, x_n) = n$

2. If X is irreducible and $\dim(X) = 0$, then $k \subseteq k(X)$ is algebraic extension, then $k(X) = k$. Thus, X is a point.

Some terminology: a curve is a variety of dimension 1, a surface is a variety of dimension 2, and an n-fold is a variety of dimension n.

Notation: If R is a finitely generated domain over k, then we write $\operatorname{tr}\deg(R) = \operatorname{tr}\deg(R_0)$.

We will state the following without proof.

Theorem 16 (Principle Ideal Theorem)**.** *If R is a finitely generated domain over k and $0 \neq f \in R$ and $P \subseteq R$ is a minimal prime, then P is a minimal prime containing f. Then* $\operatorname{tr}\deg(R/P) = \operatorname{tr}\deg(R) - 1$*.*

Geometric Statement: If X is any irreducible variety and $0 \neq f \in k[X]$ and if $Z \subseteq V(f)$ is an irreducible component then $\dim Z = \dim X - 1$.

Proof. Take $U \subseteq X$ open affine such that $U \cap Z \neq \emptyset$. Then $Z \cap U = V(P) \subseteq U$, $P \subseteq k[U]$ prime ideal. Z is a component of $V(f) \Rightarrow P$ is minimum over $(f) \subseteq k[U]$.

Thus, $\dim(Z) = \operatorname{tr}\deg(k[U]/P) = \operatorname{tr}\deg k[U] - 1 = \dim X - 1$. $\square$

Theorem 17. *Let X be an irreducible variety, and let $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$ be a maximal chain of irreducible closed subsets. Then* $\dim(X) = n$*.*

Proof. WLOG, X is affine. Take $0 \neq f \in I(X_{n-1})$. Then $X_{n-1} \subseteq V(f)$ is a component. PIT says that $\dim X_{n-1} = \dim X - 1$.

Induction implies that $\dim X_{n-1} = n - 1$, so $\dim X = n$. $\square$

Definition 40. *If X is any variety, set* $\dim(X)$ =*the supremum of all n such that $\exists$ a chain $\emptyset \neq X_1 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X$ where $X_i \subseteq$ irreducible and closed for all i.*

Exercises

1. $X = X_1 \cup \ldots \cup X_m$ and $X_i \subseteq X$ closed, then $\dim X = \max \dim(X_i)$

2. $\dim(X \times Y) = \dim X + \dim Y$

Recall: If R is a ring, then $\dim(R)$ is the supremum of all n such that $\exists$ $P_n \subsetneq P_{n-1} \subsetneq \ldots \subsetneq P_0 \subseteq R$ where P_i is a prime ideal.

Note: If X is affine then $\dim X = \dim k[X]$.

Theorem 18 (PIT For Several Equations)**.** *If X is an irreducible variety and $f_1, \ldots, f_r \in k[X]$ and $Z \subseteq V(f_1, \ldots, f_r)$ are components, then* $\dim Z \geq \dim X - r$.

Proof. Enough to show that if $W \subseteq X$ is a closed subset and each component of W has dim $\geq d$, then each component of $W \cap V(f)$ has dim $\geq d-1$ for all $f \in k[X]$.

Let $Z \subseteq W$ be a component. If $f|_Z = 0$ then $V(f) \cap Z = Z$. If $f|_Z \neq 0$ then every component of $Z \cap V(f)$ has dim$= \dim(Z) - 1 \geq d-1$.

Therefore $W \cap V(f) =$ union of finitely many irreducible closed subsets of dim $\geq d-1$. □

Lemma 4 (Prime Avoidance)**.** *X is an affine variety, $Z \subseteq X$ an irreducible closed subset and $X_1, \ldots, X_m \subseteq X$ are also irreducible closed subsets, then if $X_i \not\subseteq Z$ then $\exists f \in I(Z)$ such that $f \notin I(X_i)$.*

Proof. Induction on m.

If $m = 1$, then $X_1 \not\subseteq Z \Rightarrow I(Z) \not\subseteq I(X_1) \Rightarrow \exists f \in I(Z) \setminus I(X_1)$.

For $m \geq 2$, take $f_i \in I(Z)$ such that $f_i \notin I(X_j)$ for $j \neq i$. If any $f_i \notin I(X_j)$, then done. Take $f = f_i$.

If $f_i \in I(X_i)$ for all i, then $f = f_1 + f_2 f_3 \ldots f_m \in I(Z)$. □

Definition 41 (Codimension)**.** *If X is any variety, $Z \subseteq X$ closed and irreducible, let $X_1, \ldots, X_m$ be the components of X containing Z. Set* $\operatorname{codim}(Z; X) = \dim(X_1 \cup \ldots \cup X_m) - \dim Z$.

E.g. X is the union of a line and a plane, Z is a single point of X. Then $\operatorname{codim}(Z; X)$ is 2 if it is a point in the plane, 1 otherwise.

Theorem 19 (Reverse PIT)**.** *X affine, $Z \subseteq X$ irreducible closed and $c =$* $\operatorname{codim}(Z; X)$. *Then $\exists f_1, \ldots, f_c \in k[X]$ such that $Z \subseteq V(f_1, \ldots, f_c)$ irreducible component.*

Proof. If Z is a component of Z, then $c = 0$.

Otherwise, no components of X are contained in Z, so the lemma implies that there exists $f_1 \in k[X]$ such that $f_1 \in I(Z)$ and f_1 does not vanish on any component of X. PIT implies that $\operatorname{codim}(Z; V(f)) < c$.

Induction on c gives us that there are $f_2, \ldots, f_c \in I(Z)$ such that Z is a component of $V(f_1, \ldots, f_c)$. □

Resultants

Let K be an arbitrary field, and $f(T) = a_nT^n + \ldots + a_1T + a_0$ and $g(T) = b_mT^m + \ldots + b_1T + b_0 \in K[T]$.

Q: Do $f(T)$ and $g(T)$ have a common factor?

Set $A = \begin{bmatrix} a_n & a_2 & a_1 & a_0 & 0 \\ & a_n & a_2 & a_1 & a_0 \\ b_m & b_1 & b_0 & & \\ & b_m & b_1 & b_0 & \\ & & b_m & b_1 & b_0 \end{bmatrix}$.

Definition 42 (Resultant)**.** *We define* $\mathrm{Res}(f,g) = \det A \in K$*.*

Let $\vec{v} = (c_{m-1}, c_0, d_{n-1}, d_1, d_0) \in K^{n+m}$, then $\vec{v} \cdot A = (r_{n+m-1}, \ldots, r_1, r_0) \in K^{n+m}$. Then $(c_{m-1}^{m-1} + \ldots + c_1T + c_0)f(T) + (d_{n-1}T^{n-1} + \ldots + d_1T + d_0)g(T) = r_{m+n-1}T^{m+n-1} + \ldots + r_1T + r_0$.

Proposition 15. *Suppose $a_n \neq 0$, then* $\mathrm{Res}(f,g) \neq 0 \iff (f,g) = 1 \in K[T]$*.*

Proof. $\mathrm{Res}(f,g) = 0$ iff $\exists \vec{v} \in K^{m+n}$ such that $\vec{v} \cdot A = 0$ iff $\exists p(T), q(T)$ of deg $\leq m-1, n-1$ such that $p(T)f(T) = q(T)g(T)$, iff $(f,g) \neq 1$. □

If $f(T) = \sum_{i=0}^n a_iT^i$ then we allow formal differentiation, that is, $f'(T) = \sum_{i=1}^n ia_iT^{i-1} \in K[T]$.

Note that $(fg)' = f'g + fg'$, and similar rules still hold.

Corollary 7. *If $f(T) = a_nT^n + \ldots + a_1T + a_0$ then $f(T)$ has n different roots in $\overline{K}$ iff* $\mathrm{Res}(f,f') \neq 0$*.*

Proof. $f(T) = a_n \prod_{i=1}^r (T - \alpha_i)^{d_i}$, where $\alpha_i \neq \alpha_j$ for $i \neq j$. Then $f'(T) = a_n \sum_{i=1}^r d_i(T-\alpha_i)^{d_i-1} \prod_{j \neq i}(T-\alpha_j)^{d_j}$.

$\mathrm{Res}(f,f') = 0$ iff $(f,f') \neq 1$ iff $f'(\alpha_\ell) = 0$ for some ℓ iff $d_\ell \geq 2$ for some ℓ. □

Definition 43 (Discriminant)**.** *The discriminant of $f(T)$ is* $\mathrm{Res}(f,f')$*.*

Exercise: $a \neq 0$ and $f(T) = aT^2 + bT + c$ then discriminant$= -a(b^2 - 4ac)$.

Remark: If $\mathrm{char}(K) = 0$ and if $f(T) \in k[T]$ is an irreducible polynomial, then $(f(T), f'(T)) = 1$ so $\mathrm{Res}(f,f') \neq 0$. In $\mathrm{char}(K) = p$, then $f'(T)$ may be zero, for example $(T^p + 1)' = 0$.

Remark: $\varphi : X \to Y$ is a morphism, then $\dim(\overline{\varphi(X)}) \leq \dim(X)$. This is as $k(\overline{\varphi(X)}) \subseteq k(X)$.

Theorem 20. *$\phi : X \to Y$ is a dominant morphism of irreducible varieties, such that $k(Y) \subseteq k(X)$ is a finite extension of degree d. Suppose that* $\mathrm{char}(k) = 0$ *or $k(X)/k(Y)$ is separable. Then $\exists$ dense open $V \subseteq Y$ such that $|\phi^{-1}(y)| = d$ for all $y \in V$.*

Proof. Assume X, Y affine and $k[X] = k[Y][f]$. Let $P(T) = a_d T^d + \ldots + a_1 T + a_0 \in k(Y)[T]$ be the minimum polynomial for $f \in k(X)$ over $k(Y)$. ie $P(f) = 0 \in k(X)$.

WLOG, $a_i \in k[Y]$ for all i and we can replace Y with $D(a_d)$ and X with $\phi^{-1}(D(a_d)) \subseteq X$. We may assume that $a_d = 1$. Now $k[X] = k[Y][T]/(P(T))$. This implies that $X \simeq V(P) \subseteq Y \times \mathbb{A}^1 \xrightarrow{\pi_Y} Y$, and $\phi : X \to Y$ goes through this path.

If $(y, t) \in Y \times \mathbb{A}^1$ then we set $P_y(T) = \sum_{i=0}^{d} a_i(y) T^i \in k[T]$. $(y, t) \in X$ iff $P_y(t) = 0$. Let $\Delta = \mathrm{Res}(P, P') \in k[Y]$. $P(T)$ irreducible and $\mathrm{char}(k) = 0$ imply that $\Delta \neq 0$. Note that $\mathrm{Res}(P_y, P_y') = \Delta(y)$.

Thus, if $y \in D(\Delta)$, then $P_y(t) = 0$ has exactly d solutions.

Now the general case: X and Y are irreducible varieties and $\phi : X \to Y$ dominant. Let $V \subseteq Y$ and $U \subseteq \phi^{-1}(V) \subseteq X$ be open affines. Then $\dim(\overline{\phi(X \setminus U)}) \leq \dim(X \setminus U) < \dim(X) = \dim(Y)$. Thus $\overline{\phi(X \setminus U)} \subsetneq Y$, and so $\exists h \in k[V]$ such that $D(h) \cap \phi(X \setminus U) = \emptyset$.

We can replace X with $D(\phi^* h)$ and Y with $D(h)$. And so, WLOG, X, Y affine.

ϕ dominant implies that $k[Y] \subseteq k[X]$ and $k[X]$ generated by $f_1, \ldots, f_n$. Then $k[Y] \subseteq k[Y][f_1] \subseteq \ldots \subseteq k[Y][f_1, \ldots, f_n] = k[X]$ gives $X = X_n \to X_{n-1} \to \ldots \to X_1 \xrightarrow{\psi} Y$ a sequence of dominant maps.

Induction on n: $\exists$ a dense open $U \subseteq X_1$ such that all points of U are hit by $d_1 = [k(X) : k(X_1)]$ pts of X.

As above: $\overline{\psi(X_1 \setminus U)} \subsetneq Y$ implies $\exists h \in k[Y]$ such that $D(h) \cap \psi(X_1 \setminus U) = \emptyset$, and so $\psi^{-1}(D(h)) = D(\phi^* h) \subseteq U$. $\psi : D(\phi^* h) \to D(h)$ gives $k[D(\phi^* h)] = k[X]_{\phi^* h} = k[X]_h$. Thus, $k[Y][f_1]_h = k[Y]_h[f_1] = k[D(h)][f_1]$.

Thus, the first case implies that $\exists \emptyset \neq V \subseteq D(h) \subseteq Y$ open such that $|\psi^{-1}(y)| = [k(X_1) : k(Y)]$. Since $\psi^{-1}(D(h)) \subseteq U$ we have $|\phi^{-1}(y)| = [k(X_1) : k(Y)] \cdot d_1 = d$. $\square$

<u>Exercise</u>: $\pi_Y : X \times Y \to Y$ is an open map. That is, if $U \subseteq X \times Y$ is open then $\pi_Y(U)$ is open in Y.

Corollary 8. *$\phi : X \to Y$ is a dominant morphism of irreducible varieties. Then $\phi(X)$ contains a dense open subset of Y.*

Proof. We can assume that X, Y are affine. Choose $B = \{f_1, \ldots, f_n\} \subseteq k[X]$ such that B is a transcendence basis for $k(X)/k(Y)$.

Then $k[Y] \subseteq k[Y][f_1, \ldots, f_n] \subseteq k[X]$ gives $X \xrightarrow{\psi} Y \times \mathbb{A}^n \xrightarrow{\pi_Y} Y$ and ϕ is the composition.

The theorem says that there is a open subset $U \subseteq Y \times \mathbb{A}^n$ such that $U \subseteq \psi(X)$. As π_Y is an open mapping, $\pi_Y(U)$ is open. $\square$

Definition 44. *Let X be a variety.*

1. *$W \subseteq X$ is locally closed if $W =$ open $\cap$ closed.*

2. *$W \subseteq X$ is constructible if $W =$the union of finitely many locally closed subsets.*

<u>Example</u>: $W = D(xy) \cup \{0\} \subseteq \mathbb{A}^2$. Notice: $\phi : \mathbb{A}^2 \to \mathbb{A}^2 : (x, y) \mapsto (x^2y, xy)$, then $\phi(\mathbb{A}^2) = W$.

<u>Exercise</u>: $\phi : X \to Y$ is an arbitrary morphism of varieties, then $\phi(X)$ is constructible.

Chapter 5

Nonsingular Varieties

Local Rings

Definition 45 (Local Ring at a point)**.** *If X is an irreducible variety and $x \in X$ then $\mathscr{O}_{X,x} = \{f \in k(X) : f(x)$ defined$\}$. This is a local ring with maximal ideal $\mathfrak{m}_x = \{f \in \mathscr{O}_{X,x} : f(x) = 0\}$.*

Note: If $U \subseteq X$ is any open subset, then $k[U] = \bigcap_{x\in U} \mathscr{O}_{X,x} \subseteq k(X)$.

Let $U \subseteq X$ open affine, $x \in U$ then $M = I(\{x\}) \subseteq k[U]$. $f \in \mathscr{O}_{X,x}$ then f is defined on $D(h) \subseteq U$ for some $h \in k[U] \setminus M$, so $f = g/h^n$ where $g \in k[U]$. And so, $\mathscr{O}_{X,x} = \{g/h : g, h \in k[U], h(x) \neq 0\} = k[U]_M$.

Remark: X not irreducible implies that $\mathscr{O}_{X,x} = \varinjlim_{U\ni x} \mathscr{O}_X(U)$. If $U \subseteq X$ open affine, $x \in U$ we still have that $\mathscr{O}_{X,x} = k[U]_{I(\{x\})}$.

Note: If X is irreducible then $\dim(X) = \dim \mathscr{O}_{X,x}$ for any $x \in X$. If we let X be any variety, then $\dim(X) = \max_{x\in X} \dim \mathscr{O}_{X,x}$.

Definition 46 (Regular Local Ring)**.** *If $(R, \mathfrak{m})$ is a local ring, then $F = R/\mathfrak{m}$ is a field called the residue field of R and $\mathfrak{m}/\mathfrak{m}^2$ is an F-vector space.*

R is a regular local ring if $\dim_F(\mathfrak{m}/\mathfrak{m}^2) = \dim R$.

Exercise: If R is a Nötherian Local Ring then $\dim_F(\mathfrak{m}/\mathfrak{m}^2)$ =min number of generators for $\mathfrak{m} \geq \dim(R)$. The equality is from Nakayama, the inequality from PIT.

Definition 47. *Let X be a variety and $x \in X$.*

1. *X is nonsingular at x if $\mathscr{O}_{X,x}$ is a regular local ring.*
2. *Otherwise, X is singular at x.*

3. X is nonsingular if all points $x \in X$ are nonsingular.

Exercise: S is a commutative ring, and $M \subseteq S$ is a maximal ideal, then set $R = S_M$ a local ring. Then unique maximal ideal $\mathfrak{m} = MS_M$. Show that $S/M \simeq R/\mathfrak{m}$ and $M/M^2 = \mathfrak{m}/\mathfrak{m}^2$.

Example: Let $C = V(f) \subseteq \mathbb{A}^2$ a curve. Let $P = (0,0) \in C$. So $f = ax + by + HOT$. And $k[C] = k[x,y]/(f)$. Let $M = I(\{P\}) = (\bar{x}, \bar{y}) = (x,y)/(f) \subseteq k[C]$.

$\mathfrak{m}_P/\mathfrak{m}_P^2 = M/M^2 = (x,y)/(x^2,xy,y^2,f) = (x,y)/(x^2,xy,y^2,ax+by)$. So $P \in C$ is nonsing iff $\dim_k(M/M^2) = \dim(C) = 1$. This is iff $ax + by \neq 0 \in k[x,y]$, note that $ax + by \neq 0$ implies that $V(f)$ looks like $V(ax+by)$ close to the point.

Exercise: Find all singular points of $V(y^2 - x^3 - x^2), V(y^2 - x^3)$.

$X \subset \mathbb{A}^n$ a closed affine variety. Then $I = I(X) = (f_1, \ldots, f_t) \subseteq S = k[x_1, \ldots, x_n]$. Idea: X is nonsing at $P \in X$ iff X has a tangent space at P.

Definition 48 (Jacobi Matrix)**.** *Let $J_P = \left[\frac{\partial f_i}{\partial x_j}(P)\right]$ be a $t \times n$ matrix, we call this the Jacobi matrix.*

Note: If $\vec{v} \in k^n$ then $J_p \cdot \vec{v} \in k^t$ is the partial derivative of $(f_1, \ldots, f_t)$ at p in the direction $\vec{v}$.

$\ker(J_P) = \{\vec{v} \in k^n : J_p \cdot \vec{v} = \vec{0}\}$ is a candidate for a tangent space.

Lemma 5. *Let $P \in X \subseteq \mathbb{A}^n$, then* $\operatorname{rank}(J_p) + \dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = n$.

Proof. Set $M = I(\{P\}) \subseteq S$. Define $d : M \to k^n$ by $d(f) = (\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P))$. d is surjective as $d(x_i - p_i) = e_i$. Note: $f, g \in S$ that $d(fg) = f(P)d(g) + d(f)g(P)$. So $d(M^2) = 0$. Thus, $d : M/M^2 \to k^n$ is an isomorphism if it is injective. It is injective as the two vector spaces are of the same dimension and it is surjective.

$d(f_i) = i^{th}$ row of J_P and $d(\sum g_i f_i) = \sum g_i(P) d(f_i)$ so $d(I)$ =row span of J_P in k^n. Thus $d : I + M^2/M^2 \to$ row span of J_P is an isomorphism. Thus $\operatorname{rank}(J_p) + \dim(M/I + M^2) = \dim(M/M^2) = n$. Finally $\mathscr{O}_{X,P} = (S/I)_M$ and $\mathfrak{m}_P/\mathfrak{m}_P^2 = (M/I)/(M/I)^2 = M/I + M^2$. □

Theorem 21. *$P \in X \subseteq \mathbb{A}^n$, then* $\operatorname{rank}(J_P) \leq n - \dim(\mathscr{O}_{X,P})$ *and* $\operatorname{rank} J_P = n - \dim \mathscr{O}_{X,P} \iff P$ *nonsingular.*

Proof. $\operatorname{rank}(J_p) = n - \dim(\mathfrak{m}_P/\mathfrak{m}_P^2) \leq n - \dim \mathscr{O}_{X,P}$.

We have equality iff $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = \dim \mathscr{O}_{X,P}$. □

Example: $X = V(z^2 - x^2y^2) \subseteq \mathbb{A}^3$ where $\mathrm{char}(k) \neq 2, 3$. Then $J = [-2xy^2, -2x^2y, 3z^2]$. $p \in X$ is a nonsingular point iff $\mathrm{rank}(J_p) = 3 - x = 1$, so $X_{sing} = V(z^3 - x^2y^2, xy^2, x^2y, z^2) = V(xy, z)$.

Exercise: X is a variety and $p \in X$. Then $\mathscr{O}_{X,P}$ is a domain iff p is in only one component.

Theorem 22. *Any Nötherian regular local ring is a domain. (in fact, a UFD, and even Macaulay)*

Conclude: The points on an intersection of two components are singular.

Proposition 16. *$X_{sing} \subseteq X$ is a closed subset of X.*

Proof. Let $X = X_1 \cup \ldots \cup X_m$ be the components of X. Then $X_{sing} = \bigcup_{i=1}^m (X_i)_{sing} \cup \bigcup_{i \neq j} X_i \cap X_j$. The latter are closed, so without loss of generality, X irreducible and affine.

$X \subseteq \mathbb{A}^n$ closed, $I(X) = (f_1, \ldots, f_t) \subseteq k[x_1, \ldots, x_n]$, $P \in X_{sing}$ iff $p \in X$ and $\mathrm{rank}(J_p) < n - \dim \mathscr{O}_{X,p} = n - \dim(X)$.

Let $m_1, \ldots, m_N$ be all of the minors of size $n - \dim(X)$ in $J = [\frac{\partial f_i}{\partial x_j}]$. $X_{sing} = X \cap V(m_1, \ldots, m_N)$. $\square$

Fact: $X_{sing} \neq X$.

Lemma 6. *If $p \in X = V(g_1, \ldots, g_r) \subseteq \mathbb{A}^n$ and if $\mathrm{rank}\, J_p(g_1, \ldots, f_r) = r$ then $\mathscr{O}_{X,p}$ is regular local of dimension $n - r$.*

Proof. PIT implies that $\dim \mathscr{O}_{X,p} \geq n - r$.

$I(X) = (f_1, \ldots, f_t) \supseteq (g_1, \ldots, g_r)$. Thus, row span $J_p(f_1, \ldots, f_t) \supseteq$ row span $J_p(g_1, \ldots, g_r)$, so $r = \mathrm{rank}\, J_p(g_1, \ldots, g_r) \leq \mathrm{rank}\, J_p(f_1, \ldots, f_t) \leq n - \dim \mathscr{O}_{X,p} \leq r$. $\square$

Theorem 23 (Implicit Function Theorem)**.** *If $f_1, \ldots, f_c$ are holomorphic functions in a classical nbhd of $p \in \mathbb{C}^n$. Suppose $\det \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{1 \leq i,j \leq c} \neq 0$. Then $\exists$ holomorphic functions $w_1, \ldots, w_c$ on classical open subset of $\mathbb{C}^{n-c}$ and classical open subset $V \subseteq \mathbb{C}^n$ such that $p \in V$ and so that for all $z \in V$, $f_1(z) = \ldots = f_c(z) = 0$ iff $z - i = w_i(z_{c+1}, \ldots, z_n)$ for all $1 \leq i \leq c$.*

Theorem 24. *$X \subseteq \mathbb{C}^n$ a complex affine variety. $p \in X$ a nonsingular point. Then a classical neighborhood of p in X is holomorphic to a classical open subset of $\mathbb{C}^d$ where $d = \dim \mathscr{O}_{X,p}$.*

Proof. WLOG, X is irreducible. $I(X) = (f_1, \ldots, f_t)$ and rank $J_p(f_1, \ldots, f_t) = n - d = c$. WLOG, $\det(\frac{\partial f_i}{\partial x_j}(p)) \neq 0$. Set $Y = V(f_1, \ldots, f_c) \subseteq \mathbb{C}^n$, then rank $J_p(f_1, \ldots, f_c) = c$ implies that p is a nonsingular point of Y. So $\mathscr{O}_{Y,p}$ is regular local of dimension d.

Now, only one component of Y contains p, $p \in X$ and $X \subseteq Y$. $\dim X$ is the same as the dimension of the component of Y containing p, and as X is irreducible, X is the component of Y containing p. Then there exists open $U \subseteq \mathbb{A}^n$ such that $X \cap U = Y \cap U = V(f_1, \ldots, f_c) \cap U$. We apply the IFT, let $V \subset U$, $p \in V$, and $w_1, \ldots, w_c$ be as in the IFT.

Define $\pi : \mathbb{C}^n \to \mathbb{C}^d$ by $\pi(z) = (z_{c+1}, \ldots, z_n)$. Then $\pi : X \cap V \to \pi(X \cap V) \subseteq \mathbb{C}^d$ is an holomorphism, as we can get an inverse map $\pi^{-1} : \pi(X \cap V) \to X \cap V$ by $(z_{c+1}, \ldots, z_n) \mapsto (w_1(z), \ldots, w_c(z), z_{c+1}, \ldots, z_n)$. □

Corollary 9. *Every nonsingular complex variety variety is a complex manifold.*

If X is an affine variety, recall that pts in X are in 1-1 correspondence with max ideals in $k[X]$.

So X an irreducible variety, $p \in X$ corresponds to local rings $\mathscr{O}_{X,P} = \{f \in k(X) : f(P) \text{ defined }\}$.

Lemma 7. *X an irred var, $x, y \in X$, if $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$ then $x = y$.*

Proof. Take open affine $x \in U \subseteq X$, $y \in V \subseteq X$. X is separated, so $U \cap V$ is affine and $k[X] \otimes k[Y] \to k[U \cap V]$ is surjective, and $k[U] \subseteq \mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$ with $k[V] \subseteq \mathscr{O}_{X,y}$, thus, $k[U \cap V] \subseteq \mathscr{O}_{X,y}$.

$k[V] \subseteq k[U \cap V] \subseteq \mathscr{O}_{X,y}$ proves that $y \in U \cap V$. Then $k[U \cap V] \cap \mathfrak{m}_y$ is a max ideal in $k[U \cap V]$ so it corresponds to a point in $U \cap V$ which maps to y under $U \cap V \to V$ the inclusion.

Thus, $x, y \in U \subseteq X$. U is affine. If $x \neq y$, then there is an $f \in k[U]$ such that $f(x) \neq 0$, $f(y) = 0$. Then $\frac{1}{f} \in \mathscr{O}_{X,x}$ and $\frac{1}{f} \notin \mathscr{O}_{X,y}$. □

Chapter 6

Nonsingular Curves

Definition 49 (Curves)**.** *A curve is an irreducible variety of dimension one.*

Example: $C = \mathbb{A}^1$. $k[C] = k[t]$ and $k(C) = k(t)$. Let $0 \neq f \in k(t)$. What is the order of vanishing of f at 0?

$f = p/q$, $p, q \in k[t]$, we can write $p = t^n p_0$ and $q = t^m q_0$ with $p_0(0) \neq 0$ and $q_0(0) \neq 0$. $f = (p_0/q_0)t^{n-m}$, so $v(f) = n - m$ is the order of vanishing.

Note that $\mathscr{O}_{C,0} = k[t]_{(t)}$ with $m_0 = (t) \subseteq \mathscr{O}_{C,0}$ is a maximal ideal. Then $f = ut^{n-m}$ where $u = p_0/q_0$ is a unit in $\mathscr{O}_{C,0}$.

Definition 50 (Discrete Valuation Ring)**.** *A discrete valuation ring, or DVR, is a Nötherian regular local ring of dimension 1.*

Examples are $\mathscr{O}_{C,P}$ where C is a curve and P a nonsingular point.

Let $(R, \mathfrak{m})$ be a DVR, $\mathfrak{m} = (t)$, t is a uniformizing parameter, and $K = R_0$ is the field of fractions of R.

Claim: Any $f \in K^* = K \setminus \{0\}$ can be written $f = ut^n$ where $u \in R \setminus \mathfrak{m}$ a unit in R and $n \in \mathbb{Z}$. WLOG, $f \in R \setminus \{0\}$.

Assume that the claim is false, then choose a counterexample such that (f) is maximal. f is not a unit implies that $f \in \mathfrak{m}$, so $f = gt$ for some $g \in R$.

$(f) \subsetneq (g)$ as $(f) = (g) \Rightarrow g = hf \Rightarrow f = thf \Rightarrow th = 1$, contradiction.

So $g = ut^n$, $u \in R$ a unit implies that $f = ut^{n+1}$.

Check that if $f = ut^n$, u a unit, then u, n are unique. $n = \min\{p \in \mathbb{Z} : f \in (t^p) \subseteq K\}$.

Definition 51 (Valuation Map)**.** $v : K^* \to \mathbb{Z}$ *is a valuation map* $v(f) = n$ *if* $f = ut^n$ *with* $u \in R \setminus \mathfrak{m}$.

Note that $R = \{f \in K^* : v(f) \geq 0\} \cup \{0\}$ and $\mathfrak{m} = \{f \in K^* : v(f) > 0\} \cup \{0\}$.

Rules: $v(fg) = v(f)+v(g)$. $v(f+g) \geq \min(v(f), v(g))$ if $f, g, f+g \in K^*$. If $f = ut^n, g = vt^m$ and $n \leq m$, then $f + g = (u + vt^{m-n})t^n = u't^rt^n$.

Example: C a curve, $p \in C$ nonsing, then $R = \mathscr{O}_{C,p}$ is a DVR with $K = R_0 = k(C)$. $v_p : K(C)^* \to \mathbb{Z}$ is a valuation, $f \in k(C)^*$ $v_p(f)$ =the order of vanishing of f at p. $v_p(f) > 0$ iff $f \in \mathfrak{m}_p$, $v_p(f) = 0$ iff $f(p) \neq 0$ and $v_p(f) < 0$ iff f is not defined at p.

Lemma 8. *Let R be a DVR, $R_0 = K$ and S any ring such that $R \subseteq S \subseteq K$. Then $S = R$ or $S = K$.*

Proof. If $S \neq R$, then take $f \in S$, $f \notin R$. $\mathfrak{m} = (t)$, $f = ut^n$, $n < 0$, $S \supseteq R[f] = K$. □

Recall: A domain R is called integrally closed iff for all $f \in R_0$, f is integral over R implies that $f \in R$.

Theorem 25. *If R is any Nötherian Local Domain of dimension one, then R is regular iff R is integrally closed.*

Exercise: Prove $\Rightarrow$ (easy, as $DVR \Rightarrow PID \Rightarrow UFD \Rightarrow$ int closed, so prove the last one)

Proposition 17. *Let A be a domain. A is integrally closed iff A_P is integrally closed for all maximal ideals $P \subseteq A$.*

Proof. $\Rightarrow$: Easy

$\Leftarrow$: $A = \cap_{P \subseteq A} A_P$ over maximal ideals P. □

Note: A a f.g. domain over k, $X = \operatorname{Spec} - m(A)$, then $A = k[X] = \cap_{p \in X} \mathscr{O}_{X,p} = \cap A_p$.

Definition 52 (Dedekind Domain)**.** *A Dedekind Domain is an integrally closed Nötherian domain of dimension 1.*

Note: If X is an irred affine variety, X nonsingular curve iff $k[X]$ is a Dedekind domain.

$\mathscr{O}_{X,p}$ a DVR for all $p \in X$ iff $k[X]_p$ integrally closed for all max ideals p by the theorem, each of these is integrally closed iff $k[X]$ is, and that is the def of a Dedekind domain.

Finiteness of Integral Closure

Let R be a finitely generated domain over k, $K = R_0$, and $K \subseteq L$ a finite field extension. Then $\overline{R}$ is defined to be the integral closure of R in L. That is, $\{f \in L : f \text{ integral over } R\}$. Fact $\overline{R}$ is an integrally closed domain with field of fractions L.

Let $f \in L$. Then $f^n + a_1 f^{n-1} + \ldots + a_n = 0$ with $a_i \in K$. Take $b \in R$ such that $ba_i \in R$ for all i. Then $(bf)^n + ba_1(bf)^{n-1} + \ldots + b^n a_n = 0$. Thus $bf \in \overline{R}$. $b \in R$, so $f \in (\overline{R})_0$.

Theorem 26 (Finiteness of Integral Closure)**.** *$\overline{R}$ is a finitely generated R-module.*

In particular: $\overline{R}$ is a finitely generated K algebra.

Definition 53 (DVR of K/k)**.** *$k \subseteq K$ is a field extension, a DVR of K/k is a subring $R \subseteq K$ such that:*

1. *R is a DVR*
2. *$R_0 = K$*
3. *$k \subseteq R$.*

Let K be a function field of dimension 1 over k.

i.e., $k \subseteq K$ is finitely generated as a field extension and it has transcendence degree 1.

Q: $\exists$ a nonsingular curve C such that $k(C) = K$?

Key construction: Let $f \in K \backslash k$. Then $k(f) \subseteq K$ is a finite field extension. It is finitely generated by assumption, and $\{f\}$ must be a transcendence basis for K/k, thus it is algebraic, and so it is a finite field extension.

Set $B = \overline{k[f]} \subseteq K$.

Finiteness of integral closure says that B is a finitely generated k-algebra, and so B is a Dedekind domain with $B_0 = K$.

Proposition 18. *$X = \text{Spec}-m(B)$ is a nonsingular curve.*

1. *Points in X are in 1-1 correspondence with DVRs R of K/k such that $f \in R$.*
2. *Points in $V(f)$ correspond to the DVRs R of K/k such that $f \in \mathfrak{m}_R$.*

Proof. $X \to \{\text{DVRs } R \ni f\}$ is well-defined and injective.

Let R be a DVR of K/k with $f \in R$.

$k[f] \subseteq R \Rightarrow B \subseteq R$.

Set $M = B \cap \mathfrak{m}_R \subseteq B$ a prime ideal. Then $B_M \subseteq R \neq K \Rightarrow M \neq 0$ and so B_M is a DVR of K/k.

And so, the lemma implies that $B_M = R$.

Now we prove (b). $M \in V(f) \iff f \in M \iff f \in MB_M = \mathfrak{m}$. □

Corollary 10. *Every DVR R of K/k is the local ring of a nonsingular curve at some point. In particular, $R/\mathfrak{m}_R = K$.*

Proof. Let $f \in R \setminus k$, $B = \overline{k[f]} \subseteq K$.

Then $R = B_M$ for some maximal ideal $M \in \operatorname{Spec} -m(B)$. So $R = \mathscr{O}_{X,p}$. □

Corollary 11. *Given $f \in K^*$, there are only finitely many DVRs R of K/k such that $f \in \mathfrak{m}_R$.*

Also have finitely many R such that $f \notin R$.

Proof. WLOG, $f \notin k$. Set $B = \overline{k[f]} \subseteq K$. $\{R : f \in \mathfrak{m}_R\}$ is in correspondence with $V(f) \subseteq \operatorname{Spec} -m(B)$. PIT implies that $\dim V(f) = 0$. Thus, $V(f)$ is a finite set.

Note: $f \notin R$ iff $1/f \in \mathfrak{m}_R$. □

Definition 54. $C_K = \{DVRs\ of\ K/k\}$

Elements of C_K will be called "points" P. DVR given by P is R_P with maximal ideal $\mathfrak{m}_P$.

We define a topology on C_K to have as closed sets the finite sets and all of C_K. We also let $f \in K$ and $P \in C_K$, assume $f \in R_P$.

Definition 55. *$f(P)$ is defined to be the image of f by $R_P \to R_P/\mathfrak{m}_P = k$. i.e. $f(P) \cong f \mod \mathfrak{m}_P$.*

The regular functions on a nonempty open subset $U \subseteq C_K$ are then the set $k[U] = \cap_{p \in U} R_P \subseteq K$.

This makes C_K a SWF.

Note: If $U \subseteq C_K$ open, $f \in k[U]$ then $D(f) = \{P \in U : f(P) \neq 0\} = \{P \in U : f \notin \mathfrak{m}_P\}$ is open by the second corollary.

Example Let $K = k(t)$. The DVRs of K/k are $k[t]_{(t-a)}$ for $a \in k$ and $k[1/t]_{(1/t)}$.

Then C_K and $\mathbb{P}^1$ are in 1-1 correspondence, with $k[t]_{(t-a)}$ corresponding to $a \in \mathbb{A}^1$ and the other point corresponding to the point at infinity.

Note: If $f \in k[t]_{(t-a)}$ then $f(t) \cong f(a) \mod (t-a)$, $f(k[t]_{(t-a)}) = f(a)$.

Theorem 27. *C_K is a non-singular curve and $k(C_K) = K$.*

Proof. Take any $f \in K \setminus k$. $B = \overline{k[f]} \subseteq K$. $U = \{p \in C_K : f \in R_P\} \subseteq C_K$ is open.

We define $\phi : \operatorname{Spec}-m(B) \to U$ by $M \mapsto B_M$. The prop implies that this is bijective.

ϕ is a homeomorphism, as the closed sets are the finite sets in both. It remains to show that this is a morphism of spaces with functions.

For $V \subseteq \operatorname{Spec}-m(B)$ open, then $k[V] = \cap_{M \in V} B_M = \cap_{P \in \phi(V)} R_P = k[\phi(V)]$.

Thus, $\phi : \operatorname{Spec}-m(B) \to U$ is an isomorphism of spaces with functions.

Note: If $P \in C_K$ then $f \in R_P$ or $1/f \in R_P$. Thus, $C_k = \operatorname{Spec}-m(\overline{k[f]}) \cup \operatorname{Spec}-m(\overline{k[f^{-1}]})$. This is an open affine cover, and so C_k is a prevariety.

Let $P, Q \in C_K$. Enough to find an open affine $U \subseteq C_K$ such that $P, Q \in U$. Take $f \in R_P \setminus \mathfrak{m}_P$ and $f \notin k$. ($f = 1 + t$, $(t) = \mathfrak{m}_P$).

If $f \in R_Q$ then P, Q are both in $\operatorname{Spec}-m(\overline{k[f]})$.

Otherwise, $1/f \in R_Q$, so both are in $\operatorname{Spec}-m(\overline{k[1/f]})$.

Thus C_K is a variety. It is nonsingular and of dimension one by construction. We must show that it is irreducible.

C_K is, in fact, irreducible because its proper closed sets are finite and C_K is infinite. $\square$

Proposition 19. *Let C be an irreducible curve and $P \in C$ a nonsingular point. Let Y be any projective variety and $\phi : C \setminus \{P\} \to Y$ is any morphism of varieties. Then $\exists!$ extension $\phi : C \to Y$.*

Note: No points in Y are "missing".

Proof. $Y \subseteq \mathbb{P}^n$ closed subset. It is enough to make $\phi : C \to \mathbb{P}^n$.

WLOG $\phi(C \setminus \{p\}) \not\subseteq V_+(x_i)$ for all i. Set $U = D(x_0, x_1, \ldots, x_n) \subseteq \mathbb{P}^n$. $\phi(C \setminus \{p\}) \cap U \neq \emptyset$.

Set $f_{ij} = x_i/x_j \circ \phi \in k(C)$. Defined on $\phi^{-1}(U) \neq \emptyset$.

$v_P : k(C)^* \to \mathbb{Z}$ is the valuation given by $\mathscr{O}_{C,P}$.

Set $r_i = v_P(f_{i0})$ for $0 \leq i \leq n$. Choose j such that r_j minimal.

As $\frac{x_i}{x_j} = \frac{x_i/x_0}{x_j/x_0}$, $f_{ij} = f_{i0}/f_{j0}$, so $v_P(f_{ij}) = r_i - r_j \geq 0$. Thus, $f_{ij} \in \mathscr{O}_{C,P}$ for all i. Note that if $Q \in \phi^{-1}(U)$, then $\phi(Q) = (f_{0j}(Q) : f_{1j}(Q) : \ldots : f_{jj}(Q) = 1 : \ldots : f_{nj}(Q))$.

We can then extend ϕ to P by this expression. Then ϕ is a morphism on $\phi^{-1}(U) \cup \{p\}$ and so $\phi : C \to \mathbb{P}^n$ is a morphism. □

Lemma 9. *$R \subseteq K$ is a local ring, $k \subseteq R$, R is not a field. Then R is contained in some discrete valuation ring of K/k.*

Proof. Set $B = \overline{R} \subseteq K$. Lying over implies that there exists some maximal ideal $M \subseteq B$ such that $M \cap R = \mathfrak{m}_R$.

Claim: B_M is a DVR of K/k.

Let $0 \neq f \in \mathfrak{m}_R$. $S = \overline{k[f]}$ is a Dedekind domain, $S \subseteq B$. $\tilde{M} = M \cap S$ is a maximal ideal of S.

Thus $S_{\tilde{M}}$ is a DVR of K/k. $S_{\tilde{M}} \subseteq B_M \subsetneq K$ and a lemma from before says that we have equality of $S_{\tilde{M}}$ and B_M. □

Theorem 28. *C_K is a projective curve.*

Proof. Let $f \in K \setminus k$. $U = \operatorname{Spec}-m(\overline{k[f]})$, $V = \operatorname{Spec}-m(\overline{k[f^{-1}]})$, and $C_K = U \cup V$ an open affine cover.

$U \subseteq \mathbb{A}^N$ closed. $\overline{U} \subseteq \mathbb{P}^N$ projective closure.

The proposition implies that the inclusion $U \to \overline{U}$ extends to a morphism $\varphi_1 : C_K \to \overline{U}$.

Similarly, we take $\overline{V}$ to be the projective closure of V. Then $V \to \overline{V}$ extends to $\varphi_2 : C_K \to \overline{V}$.

We now define $\varphi : C_k \to \overline{U} \times \overline{V}$ by $\varphi(P) = (\varphi_1(P), \varphi_2(P))$. Set $Y = \overline{\varphi(C_K)} \subseteq \overline{U} \times \overline{V}$.

Y is a projective variety.

Claim: $\varphi : C_K \to Y$ is an isomorphism.

Note: $\varphi(U) \subseteq U \times \overline{V}$ is a closed subset. Let $\psi = \varphi_2 \times \mathrm{id} : U \times \overline{V} \to \overline{V} \times \overline{V}$. $\varphi(U) = \{(u, v) \in U \times \overline{V} : \varphi_2(u) = v\} = \psi^{-1}(\Delta_{\overline{V}})$ closed. Thus, $\varphi(U) = Y \cap (U \times \overline{V})$, which implies that $\varphi : U \to Y \cap (U \times \overline{V})$ is bijective, isomorphism.

So $\pi_U : Y \cap (U \times \overline{V}) \to U$ is the inverse.

Similarly, $\varphi : V \to Y \cap (\overline{U} \times V)$ is an isomorphism.

Note: $k(Y) = k(C_K) = K$, and forall $P \in C_K$, $\mathscr{O}_{Y,\varphi(P)} = R_P \subseteq K$. Thus, φ is injective.

For surjective, let $y \in Y$. Then $k \subseteq \mathscr{O}_{Y,y} \subseteq K$ is a local ring. By the lemma, $\mathscr{O}_{Y,y} \subseteq R_P$ for some $P \in C_K$. $\mathscr{O}_{Y,y} \subseteq R_P = \mathscr{O}_{Y,\varphi(P)}$ so $y = \varphi(P)$. □

Corollary 12. *Any curve is birational to some nonsingular projective curve.*

Corollary 13. *X is any nonsingular curve, then $X \cong$ some open subset of C_K, $K = k(X)$.*

Proof. $\varphi : X \to C_K$ by $\varphi(x) = P$ where $P \in C_K$ such that $R_P = \mathscr{O}_{X,x} \subseteq K$.

Injectivity is clear. Claim: $\varphi(X) \subseteq C_K$ is open. Take $U \subseteq X$ open affine. $k[U]$ is generated by $f_1, \ldots, f_n$. $P \in \varphi(U)$ iff $k[U] \subseteq R_P$, iff $f_i \in R_P$ for all i.

Thus, $\varphi(U) = \cap_{i=1}^n \operatorname{Spec} - m(\overline{k[f_i]})$ open. Thus $\varphi(X) \subseteq C_K$ is open.

$\varphi : X \to \varphi(X)$ is a homeomorphism. To check that φ is an isomorphism, then $U \subseteq X$ open gives $k[U] \cap_{x \in U} \mathscr{O}_{X,x} = \cap_{P \in \varphi(U)} R_{\varphi(x)} = k[\varphi(U)]$. ∎

Exercise: Two nonsingular projective curves are isomorphic iff they have the same function field.

Chapter 7

Degree of Projective Varieties

Bezout: $f_1, \dots, f_n \in k[x_0, \dots, x_n]$ homogeneous of degrees $d_1, \dots, d_n$. Then $V_+(f_1, \dots, f_n)$ has cardinality at most $d_1 \dots d_n$ or is infinite. If it is finite and counted with multiplicity, then it is equal to $d_1 \dots d_n$.

Classical Definition: $X \subseteq \mathbb{P}^n$ closed, then $\deg(X) = \#(X \cap V)$ where $V \subseteq \mathbb{P}^n$ is a linear subspace with $\dim V + \dim X = n$.

e.g. $f \in S = k[x_0, \dots, x_n]$ a square-free homogeneous polynomial. Then $\#(V_+(f) \cap$ general line$) = \deg f$.

Warning: $V_+(xz - y^2)$, $V_+(x) \subseteq \mathbb{P}^2$. These are isomorphic but have different degrees. So degree is not a property of a projective variety, but rather one of the embedding into projective space.

Example: $f \in S$ is square-free, $\deg f = d$. Then $X = V_+(f)$, $I(X) = (f)$, $R = S/(f)$ is the projective coordinate ring.

Note: $\dim_k(S_m) = \binom{m+n}{n}$, where S_m is all forms of degree m. This is $\frac{1}{n!}(m+n)(m+n-1)\dots(m+1)$, which is actually a polynomial in m of degree n with lead coefficient $\frac{1}{n!}$.

Consider $0 \to S \xrightarrow{f} S \to R \to 0$ implies that $0 \to S_{m-d} \to S_m \to R_m \to 0$ is exact. Then $\dim R_m = \dim S_m - \dim S_{m-d}$, which is $\binom{m+n}{n} - \binom{m+n-d}{n}$, which is $\frac{1}{n!}(m+n)\dots(m+1) - \frac{1}{n!}(m+n-d)\dots(m+1-d)$. Which is a polynomial of degree $n-1$.

This has lead coefficient $\frac{1}{n!}\left(\sum i - \sum (i-d)\right) = \frac{nd}{n!} = \frac{d}{(n-1)!}$.

Recall: A graded S-module is a module M with a decomposition $M =$

$\oplus_{d\in\mathbb{Z}} M_d$ as abelian group such that $S_m M_d \subset M_{m+d}$.

Definition 56. $\operatorname{Ann}(M) = \{f \in S : fM = 0\}$ *is a homogeneous ideal,* $\operatorname{Supp}(M) = V_+(\operatorname{Ann}(M)) \subseteq \mathbb{P}^n$.

Reason for Supp is $x \in \mathbb{P}^n$, $P = I(\{x\}) \subseteq S$ is a homogeneous prime ideal. $M_P \neq 0$ iff $P \supseteq \operatorname{Ann}(M)$, iff $x \in \operatorname{Supp}(M)$.

Example: $X \subseteq \mathbb{P}^n$ closed, then $\operatorname{Ann}(S/I(X)) = I(X)$, so $\operatorname{Supp}(S/I(X)) = V_+(I(X)) = X$.

Note: M_d is a k-vector space for all $d \in \mathbb{Z}$ because $S_0 M_d \subseteq M_d$. If M is a finitely generated graded S-module, then $\dim_k M_d < \infty$ forall d.

Definition 57 (Hilbert Function)**.** $\mathscr{H}_M(d) = \dim_k M_d$ *is the Hilbert function of* M*.*

Theorem 29. *If M is a finitely generated graded S-module then $\exists! P_M(z) \in \mathbb{Q}[z]$ such that $P_M(d) = \dim_k(M_d)$ for all d sufficiently large. We call $P_M(z)$ the Hilbert Polynomial.*

Definition 58 (Numerical Polynomial)**.** *$P(x) \in \mathbb{Q}[z]$ is a numerical polynomial if $P(d) \in \mathbb{Z}$ for all d sufficiently large in $\mathbb{Z}$.*

<u>Example</u>: $\binom{z}{m} = \frac{1}{m!} z(z-1)\dots(z-m+1)$.

<u>Note</u>: $\left\{\binom{z}{m} : m \in \mathbb{N}\right\}$ is a basis over $\mathbb{Q}$ for $\mathbb{Q}[z]$.

Lemma 10. $P(z) = c_0 + c_1\binom{z}{1} + \dots + c_r\binom{z}{r} \in \mathbb{Q}[z]$, $c_i \in \mathbb{Q}$. *Then TFAE*

1. *$P(z)$ is a numerical polynomial.*
2. *$P(d) \in \mathbb{Z}$ for all $d \in \mathbb{Z}$.*
3. *$c_i \in \mathbb{Z}$.*

Proof. $3 \Rightarrow 2 \Rightarrow 1$ are easy, so we need $1 \Rightarrow 3$.

We know that $\binom{z+1}{m} - \binom{z}{m} = \binom{z}{m-1}$. Thus $P(z+1) - P(z) = c_1 + c_2\binom{z}{1} + \dots + c_r\binom{z}{r-1}$.

We perform induction on r: $P(z)$ is numeric, then $P(z+1) - P(z)$ is also numeric. Thus, $c_1, \dots, c_r \in \mathbb{Z}$, and so c_0 must also be an integer. $\square$

Theorem 30 (Affine Dimension Theorem)**.** *$X, Y \subseteq \mathbb{A}^n$ closed and irreducible, then $Z \subseteq X \cap Y$ component has $\dim Z \geq \dim X + \dim Y - n$.*

Proof. $X \cap Y = (X \times Y) \cap \Delta_{\mathbb{A}^n} = (X \times Y) \cap V(x_1 - y_1, \ldots, x_n - y_n) \subseteq \mathbb{A}^n \times \mathbb{A}^n$, use PIT. □

WARNING: Does not prevent $X \cap Y = \emptyset$.

Theorem 31 (Projective Dimension Theorem)**.** *$X, Y \subseteq \mathbb{P}^n$ closed irreducible, $Z \subseteq X \cap Y$ a component, then $\dim Z \geq \dim X + \dim Y - n$. If $\dim X + \dim Y - n$ is nonnegative, then $X \cap Y$ is nonempty.*

Proof. First statement follows from ADT.

Set $s = \dim X, t = \dim Y$. $C(X) = \overline{\pi^{-1}(X)} \subseteq \mathbb{A}^{n+1}$, then $\dim C(X) = s + 1$, $\dim C(Y) = t + 1$.

Every component of $C(X) \cap C(Y)$ has dimension $\geq s + 1 + t + 1 - n - 1 = s + t - n + 1 \geq 1$, and $0 \in C(X) \cap C(Y)$, so such a component exists. □

Definition 59 (Twisted Module)**.** *Let $\ell \in \mathbb{Z}$, $M(\ell)$ is the twisted module given by $M(\ell)_d = M_{\ell+d}$. i.e., we are shifting the grading, but doing nothing else.*

Definition 60 (Homogeneous Homomorphism)**.** *A homomorphism $\varphi : M \to N$ of graded S-modules is homogeneous if $\varphi(M_d) \subseteq M_d$ for all d.*

Definition 61 (Homogeneous Submodule)**.** *A submodule $N \subseteq M$ is homogeneous if $N = \oplus_{d \in \mathbb{Z}}(N \cap M_d)$.*

This implies that N and $M/N = \oplus M_d/(N \cap M_d)$ are graded and $0 \to N \to M \to M/N \to 0$ is a short exact sequence of homogeneous homomorphisms.

Note: Let $m \in M$ be homogeneous of degree ℓ, that is, $m \in M_\ell$, then $I = \operatorname{Ann}(m) \subseteq S$ is a homogeneous ideal. Set $N = S \cdot m \subseteq M$, then $0 \to I \to S \to N \to 0$ is a short exact sequence, but it is not quite of homogeneous maps, unless we change S, I to $S(-\ell), I(-\ell)$. Thus, $(S/I)(-\ell)$ is isomorphic to $N = S \cdot m \subseteq M$.

Exercise: M f.g. module over a Nötherian ring, then M is a Nötherian module.

Lemma 11. *M a f.g. graded module over a Nötherian graded ring S, then $\exists$ filtration $0 = M^0 \subseteq \ldots \subseteq M^r = M$ such that $M^i \subseteq M$ is a homogeneous submodule and $M^i/M^{i-1} \simeq S/P_i(\ell_i)$ where P_i is a homogeneous prime ideal and $\ell_i \in \mathbb{Z}$.*

Proof. Let $N \subseteq M$ be maximal homogeneous submodule such that the lemma is true for N.

Claim: $N = M$. Else $M'' = M/N \neq 0$, Take $0 \neq m \in M''$ homogeneous such that $\mathrm{Ann}(m) \subseteq S$ is as large as possible.

Claim: $P = \mathrm{Ann}(m)$ prime ideal. $P \neq S$. Let $f, g \in S \setminus P$. Enough to show that $fg \notin P$. Note: $gm \neq 0$ and $\mathrm{Ann}(gm) \supsetneq \mathrm{Ann}(m)$, thus $\mathrm{Ann}(gm) = \mathrm{Ann}(m)$. So we must have $fg \notin \mathrm{Ann}(m)$, else $f \in \mathrm{Ann}(gm)$.

Thus, $Sm \simeq (S/P)(-\ell)$, $\ell = \deg m$. So $M \to M/N = M'' \supseteq Sm$, $\tilde{N} \subseteq M$ is the inverse image of Sm, and $N \subsetneq \tilde{N}$, and the lemma is true for $\tilde{N}$. □

Definition 62 (Eventually Polynomial)**.** $f : \mathbb{Z} \to \mathbb{Z}$ *is eventually polynomial if* $\exists P(z) \in \mathbb{Q}[z]$ *such that* $f(n) = P(n)$ *for all* $n >> 0$*.*

Set $\Delta f(n) = f(n+1) - f(n)$

Lemma 12. f *is eventually polynomial of degree* r *iff* Δf *is eventually polynomial of degree* $r-1$*.*

Proof. $\Rightarrow$: Obvious.

$\Leftarrow$: Assume $\Delta f(n) = Q(n)$ for all $n >> 0$ where $Q(z) = c_1 + c_2 \binom{z}{1} + \ldots + c_r \binom{z}{r-1}$. Set $P(z) = c_1 \binom{z}{1} + \ldots + c_r \binom{z}{r}$.

Then $\Delta P = Q$, so $\Delta(f-P)(n) = 0$ for all $n >> 0$. Thus $f(n) - P(n) = c_0$ a constant for $n >> 0$. □

Set $S = k[x_0, \ldots, x_n]$, M a f.g. graded S-module.

<u>Hilbert Function</u>: $H_M(d) = \dim_k(M_d)$, and $\mathrm{Supp}(M) = V_+(\mathrm{Ann}(M)) \subseteq \mathbb{P}^n$.

Note: If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of graded S-modules, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

$\supseteq$: $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(M') \cap \mathrm{Ann}(M'')$.

$\subseteq$: Let $x \in \mathbb{P}^n$. If $x \notin \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$, then $\exists f \in \mathrm{Ann}(M')$ such that $f(x) \neq 0$ and $\exists g \in \mathrm{Ann}(M'')$ such that $g(x) \neq 0$, then $fg \in \mathrm{Ann}(M)$, but $fg(x) \neq 0$.

$\mathrm{Ann}(M')\,\mathrm{Ann}(M'') \subseteq \mathrm{Ann}(M) \subseteq \mathrm{Ann}(M') \cap \mathrm{Ann}(M'')$.

Theorem 32. M *f.g. graded module over* $S = k[x_0, \ldots, x_n]$ *then* $H_M(d)$ *is eventually equal to a polynomial* $P_M(d)$ *of degree* $\dim(\mathrm{Supp}(M))$*.*

Proof. $\exists$ filtration $0 = M^0 \subset \ldots \subset M^r = M$. such that $M^i/M^{i-1} \simeq (S/P_i)(\ell_i)$ where $P_i \subseteq S$ is a homogeneous prime.

Note: $H_M(d) = \sum_{i=1}^r H_{M^i/M^{i-1}}(d) = \sum_{i=1}^r H_{S/P_i}(d+\ell_i)$.

$\mathrm{Supp}(M) = \cup_{i=1}^r \mathrm{Supp}(M^i/M^{i-1}) = \cup_{i=1}^r V_+(P_i)$. WLOG, $M = S/P$, P a homogeneous prime. Induction of $V_+(P)$:

If $P = (x_0, \ldots, x_n)$, then the theorem is true if we take $\dim \emptyset = \deg 0 = -1$.

Otherwise, some $x_i \notin P$. Set $I = P + (x_i)$. Then $V_+(I) \subsetneq V_+(P)$, so $\dim V_+(I) = \dim V_+(P) - 1$ by the projective dimension theorem.

By induction, $H_{S/I}(d)$ is eventually polynomial and $\deg(P_{S/I}) = \dim V_+(P) - 1$

$0 \to (S/P)(-1) \to S/P \to S/I \to 0$, so $\Delta H_{S/P}(d-1) = H_{S/P}(d) - H_{S/P}(d-1) = H_{S/I}(d)$, so $H_{S/P}(d)$ is eventually polynomial of degree equal to $\dim V_+(P)$. $\square$

Note: $P_M(z) = c_0 + c_1 \binom{z}{1} + \ldots + c_r \binom{z}{r} \in \mathbb{Q}[z]$ is a numeric polynomial, so $c_i \in \mathbb{Z}$.

$r = \dim \mathrm{Supp}(M)$, so $r!(\text{lead coef of } P_M(z)) \in \mathbb{Z} \geq 0$

Definition 63 (Hilbert Polynomial of a Variety)**.** *Let $X \subseteq \mathbb{P}^n$ be a closed subvariety of dimension r, then $P_X(z) = P_{S/I(X)}(z)$ is the Hilbert Polynomial for X.*

We now define $\deg(X) = r!(\text{lead coef of } P_X(z))$.

<u>Examples</u>

1. $\deg V_+(f) = \deg f$
2. $V \subseteq \mathbb{P}^n$ linear subspace. WLOG, $V = V_+S(x_{r+1}, \ldots, x_n)$. $S/I(V) \simeq k[x_0, \ldots, x_r]$, so $P_{S/I(X)}(d) = \binom{r+d}{r}$, so lead coef is $1/r!$, so we get degree 1.

Proposition 20. *$X_1, X_2 \subseteq \mathbb{P}^n$ closed, $\dim X_1 = \dim X_2 = r$, no components in common, then $\deg(X_1 \cup X_2) = \deg X_1 + \deg X_2$*

Proof. $I_1 = I(X_1)$, $X_2 = I(X_2)$.

$I(X_1 \cup X_2) = I_1 \cap I_2$, so $0 \to S/(I_1 \cap I_2) \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0$.

The first takes $f \mapsto (f, f)$ and the second takes $(f, g) \mapsto f - g$. They are injective and surjective, and so we see this this is a short exact sequence. Thus $P_{X_1}(d) + P_{X_2}(d) = P_{X_1 \cup X_2}(d) + P_{S/(I_1+I_2)}(d)$, so $\mathrm{Supp}(S/(I_1 + I_2)) = V_+(I_1 + I_2) = X_1 \cap X_2$. $\dim < r$, so $\mathrm{LC}(P_{X_1}) + \mathrm{LC}(P_{X_2}) = \mathrm{LC}(P_{X_1 \cup X_2})$. $\square$

Corollary 14. *If $X \subseteq \mathbb{P}^n$ has dim zero, then $\deg(X) =$the number of points in X.*

Definition 64 (Simple Module)**.** *If R is a ring and M an R-module, then M is simple if M has no nontrivial submodules and $M \neq 0$. This is equivalent to $M \simeq R/P$ where $P \subseteq R$ is a maximal ideal.*

Definition 65 (Decomposition Series)**.** *A decomposition series for M is a filtration $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_r = M$ such that M_i/M_{i-1} is simple for all i.*

Definition 66 (Artinian Module)**.** *If there exists a decomposition series, then M is an Artinian module and we define $\mathrm{length}(M) = r$.*

Assume that R is Nötherian and that M is a finitely generated R-module. Then there exists a filtration $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_r = M$. such that M_i/M_{i-1} is isomorphic to R/P_i where $P_i \subseteq R$ is maximal.

Note: $\mathrm{Ann}(M) \subseteq P_i$ for all i.

Lemma 13. *If $P \subseteq R$ is a minimal prime over $\mathrm{Ann}(M)$ then M_P is an Artinian R_P-module which has $\mathrm{length}_{R_P}(M_P) = |\{i : P_i = P\}|$ in the filtration of M.*

Proof. $0 = (M_0)_P \subseteq (M_1)_P \subseteq \ldots \subseteq (M_r)_P = M_P$, and $(M_i)_P/(M_{i-1})_P = (M_i/M_{i-1})_P = (R/P_i)_P$.

If $P = P_i$, then we get R_P/PR_P, else we get 0. $\square$

$X \subseteq \mathbb{P}^n$ is a closed subvariety, $S = k[x_0, \ldots, x_n]$, then $\dim_k(S/I(X))_d = P_X(d)$ for all $d >> 0$. $\dim(X) = \deg(P_X)$ and $\deg(X) = (\dim X)!\,\mathrm{LC}(P_X)$

Assume $X \subseteq \mathbb{P}^n$ has pure dimension r. That is, all components of X have dimension r.

$Y = V_+(f) \subseteq \mathbb{P}^n$ a hypersurface such that no component of X is contained in Y. Also assume that $f \in S$ is square-free.

Let $Z \subseteq X \cap Y$ be a component. Then $\dim Z = r - 1$. Set $M = S/(I(X) + (f))$. $\mathrm{Supp}(M) = X \cap Y$, do if $P = I(Z)$, then P is minimal prime over $\mathrm{Ann}(M)$.

Definition 67 (Intersection Multiplicity)**.** $I(X \cdot Y; Z) = \text{length}_{S_P}(M_P)$

Theorem 33. $\deg(X)\deg(Y) = \sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z)\deg(Z)$.

Proof. Set $d = \deg(X)$, $e = \deg(Y)$.

Then we have a short exact sequence $0 \to S/I(X) \xrightarrow{f} S/I(X) \to M \to 0$.

This gives us $0 \to (S/I(X))_{\ell - e} \to (S/(I(X)))_\ell \to M_\ell \to 0$ will be a short exact sequence for each ℓ, where this is breaking it into homogeneous parts.

Thus $P_M(\ell) = P_X(\ell) - P_X(\ell - e)$. Thus $\text{LC}(P_M) = r \cdot e \cdot \text{LC}(P_X) = red/r! = \frac{de}{(r-1)!}$.

Take filtration $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_t = M$, $M_i \subseteq M$ is homogeneous and M_i/M_{i-1} is isomorphic to S/P_i where $P_i \subseteq P$ is a homogeneous prime ideal.

So $P_M(\ell) = \sum_{i=1}^t P_{M_i/M_{i-1}}(\ell) = \sum_{Q \subseteq S \texttt{ hom prime}} P_{S/Q}(\ell)|\{i : Q = P_i\}|$, and from this we get $\sum_{Z \subseteq X \cap Y; component} |\{i : P_i = I(Z)\}| P_Z(\ell) + LOT$. So we have $\text{LC}(P_M) = \frac{1}{(r-1)!}\sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z)\deg(Z)$. □

Corollary 15 (Bezout's Theorem)**.** *Let $X, Y \subseteq \mathbb{P}^2$ be curves of degree d and e such that $X \cap Y$ is a finite set, then $\sum_{P \in X \cap Y} I(X \cdot Y; P) = de$.*

Exercise: $P \in X \cap Y \subseteq \mathbb{P}^2$ then $I(X \cdot Y; P) = 1 \iff P$ is a nonsingular point of both X and Y and X and Y have different tangent directions at P.

Bezout's Theorem for $\mathbb{P}^n$

Idea: If $X \subseteq \mathbb{P}^n$ is closed and irreducible and $Y \subseteq \mathbb{P}^n$ is a hypersurface, then $[X] \cdot [Y] = \sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z) \cdot [Z]$.

Suppose that $Y_1, Y_2, \ldots, Y_n \subseteq \mathbb{P}^n$ are hypersurfaces such that their intersection is finite. Then $(\ldots([\mathbb{P}^n] \cdot [Y_1]) \cdot [Y_2] \ldots) \cdot [Y_n] = \sum_{i=1}^N c_i[P_i]$ where $Y_1 \cap \ldots \cap Y_n = \{P_1, \ldots, P_N\}$. Then $\sum c_i = \prod_{j=1}^n \deg(Y_j)$.

In fact: $(\ldots([\mathbb{P}^n] \cdot [Y_1]) \ldots) \cdot [Y_m] = \sum_{i=1}^{N_m} c_i^{(m)}[Z_i]$ where $Y_1 \cap \ldots \cap Y_m = Z_1 \cup \ldots \cup Z_{N_m}$ components, then $\prod_{j=1}^m \deg(Y_j) = \sum_{i=1}^{N_m} c_i^{(m)} \deg(Z_i)$.

Fact: $c_i, c_i^{(m)}$ do not depend on the order of multiplication.

Corollary 16. *$Y_1 \cap \ldots \cap Y_n$ finite implies that $|Y_1 \cap \ldots \cap Y_n| \leq \prod \deg(Y_j)$.*

Useful Fact: $X_1, \ldots, X_m \subseteq \mathbb{P}^n$ irreducible closed, $d \in \mathbb{N}$ then there exists irreducible hypersurface $Y \subseteq \mathbb{P}^n$ of degree d such that $X_i \not\subset Y$ for all i.

$v_d : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$ the veronese map, then $Y = \mathbb{P}^n \cap H$, $H \subseteq \mathbb{P}^N$ hyperplane.

Intrinsic: $\{$hypersurfaces of degree d in $\mathbb{P}^n\} \leftrightarrow \mathbb{P}(S_d)-$nonsquarefree by $V_+(f) \leftrightarrow [f]$

$\{$irreducible hypersurfaces$\} \leftrightarrow \mathbb{P}(S_d) \setminus \cup_{p+q=d} \varphi_{p,q}(\mathbb{P}(S_p) \times \mathbb{P}(S_q))$ where $\varphi_{p,q} : \mathbb{P}(S_p) \times \mathbb{P}(S_q) \to \mathbb{P}(S_{p+q}) : [f] \times [g] \mapsto [fg]$.

$\binom{n+p}{n} + \binom{n+q}{n} \leq \binom{n+p+q}{n}$ so $U \subseteq \mathbb{P}(S_d)$ is a dense open subset.

Now, if $X \subseteq \mathbb{P}^n$ is closed, then $X \subseteq V_+(f)$ iff $f \in I(X)_d$. Thus

$\{$hypersurfaces$\not\supseteq X\} \leftrightarrow W_X = \mathbb{P}(S_d) \setminus \mathbb{P}(I(X)_d)$. Now, we can conclude that $\{$irreducible hypersurfaces Y in $\mathbb{P}^n$ of degree d such that $X_i \not\subseteq Y$ for all i correspond to $U \cap W_{X_1} \cap \ldots \cap W_{X_m} \subseteq \mathbb{P}(S_d)$ is still a dense open subset.

Chapter 8

Sheaves

Definition 68 (Presheaf)**.** *Let X be a topological space. A presheaf $\mathscr{F}$ of abelian groups on X is an assignment $U \mapsto \mathscr{F}(U)$ of an abelian group $\mathscr{F}(U)$ to each open subseteq U of X plus group homomorphisms $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ whenever $V \subseteq U \subseteq X$ open with*

1 $\rho_{UU} = \mathrm{id}$

2 $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ *when* $W \subseteq V \subseteq U$

Notation: elements of $\mathscr{F}(U)$ are called sections of $\mathscr{F}$ over U, ρ_{UV} are called restriction maps and $s \in \mathscr{F}(U)$, $V \subseteq U$, then $s|_V = \rho_{UV}(s)$. Also, say $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$.

Definition 69 (Sheaf)**.** *A sheaf is a presheaf $\mathscr{F}$ such that*

3 *If $s \in \mathscr{F}(U)$ and $U = \cup V_i$ an open cover, then $s|_{V_i} = 0$ iff $s = 0$.*

4 *If $U = \cup V_i$ an open cover and have sections $s_i \in \mathscr{F}(V_i)$ and $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.*

Note: The section s of axiom 4 is unique by axiom 3.
Remark: We can easily define sheaves of rings, sets, modules, etc.
Examples:

1. X is a SWF, then we can define $\mathscr{O}_X$ of k-algebras by $\mathscr{O}_X(U) = k[U]$. This sheaf is called the structure sheaf of X.

2. Let M be a manifold, $\mathscr{O}_M(U) = \{C^\infty\ f : U \to \mathbb{R}\}$. Then there is $\pi : TM \to M$ the tangent bundle, $\pi^{-1}(x) = T_xM$. Then $TM \leftrightarrow$ a sheaf $\mathscr{T}$ of $\mathscr{O}_M$-modules. $\mathscr{T}(U) = \{s : U \to TM$ such that $\pi(s(x)) = x$ for all $x \in U\}$. $\mathscr{T}(U)$ is a $\mathscr{O}_M(U)$-module.

Definition 70 (Morphism of Sheaves)**.** *A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of (pre)sheaves consists of homomorphisms $\varphi_U : \mathscr{F} \to \mathscr{G}$ for all $U \subseteq X$ open such that $s \in \mathscr{F}$, $V \subseteq U$ open, then $\varphi_U(s)|_V = \varphi_V(s|_V) \in \mathscr{G}(V)$ and*

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r] \arrow[d, "\varphi_V"] & \mathscr{G}(U) \arrow[d, "\varphi_U"] \\ \mathscr{F}(V) \arrow[r] & \mathscr{F}(U) \end{tikzcd}$$

If $\varphi : \mathscr{F} \to \mathscr{G}$ and $\psi : \mathscr{G} \to \mathscr{H}$ are morphisms, we define $\psi \circ \varphi : \mathscr{F} \to \mathscr{H}$ by $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$. An isomorphism is a morphism with an inverse morphism.

Definition 71 (The stalk of $\mathscr{F}$ at x)**.** *If $\mathscr{F}$ is a presheaf on X, $x \in X$ then the stalk of $\mathscr{F}$ at x is $\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$. This means*

1. *$\mathscr{F}_x$ is an abelian group (or whatever)*

2. *If $x \in U$ we have homomorphism $\rho_{Ux} : \mathscr{F}(U) \to \mathscr{F}_x$, if $x \in V \subseteq U$ then $\rho_{Ux} = \rho_{Vx} \circ \rho_{UV}$.*

3. *If G is any abelian group with homomorphisms $\Theta_U : \mathscr{F}(U) \to G$ such that $\Theta_U = \Theta_V \circ \rho_{UV}$ for all $x \in V \subseteq U$, then there exists a unique group homomorphism $\Theta_x : \mathscr{F}_x \to G$ such that $\Theta_U = \Theta_x \circ \rho_{Ux}$*

<ins>Example</ins>: X a variety, $x \in X$. Then $\mathscr{O}_{X,x} = \varinjlim_{U \ni x} \mathscr{O}_U$ =local ring of X at x.

<ins>Construction</ins> $\mathscr{F}_x = \left(\bigoplus_{U \ni x} \mathscr{F}(U)\right) / \langle (0, \ldots, 0, s, 0, \ldots, 0, -s|_V, 0, \ldots, 0) : \forall s \in \mathscr{F}(U), x \in V \subseteq U \rangle$.

Notation: If $s \in \mathscr{F}_U$ and $x \in U$ write $s_x = \rho_{Ux}(s) \in \mathscr{F}_x$.

<ins>Exercise</ins> Let $\mathscr{F}$ be a presheaf.

1. All elements of $\mathscr{F}_x$ can be written as s_x for some $s \in \mathscr{F}(U)$, $x \in U$.

2. $s \in \mathscr{F}(U)$, $x \in U$, $s_x = 0 \in \mathscr{F}_x$ iff $s|_V = 0 \in \mathscr{F}(V)$ for $x \in V \subset U$.

Exercise: $\mathscr{F}$ is a sheaf. $s \in \mathscr{F}(U)$. $s = 0 \iff s_x = 0$ for all $x \in U$.

Note: A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ gives a homomorphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$, $\varphi_x(s_x) = \varphi_U(s)_x$, $s \in \mathscr{F}(U), x \in U$.

$x \in U$:

$$\begin{tikzcd} \mathscr{G}_x \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}_x \arrow[r, dashed] & \mathscr{F}(U) \end{tikzcd}$$

Proposition 21. *$\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. φ is an isomorphism iff $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are isomorphisms for all $x \in X$.*

Proof. $\Rightarrow$: Clear

$\Leftarrow$: We must show that $\varphi : \mathscr{F}(U) \to \mathscr{G}(U)$ are isomorphisms. φ_U is injective as $s \in \mathscr{F}(U)$, if $\varphi_U(s) = 0 \in \mathscr{G}(U)$ then $\varphi_x(s_x) = \varphi_U(s)_x = 0$, but φ_x is injective, and so $s_x = 0$ for all $x \in U$, so $s = 0$.

To see that φ_U is surjective, take $t \in \mathscr{G}(U)$. φ_x surjective implies that $t_x = \varphi_x(s(x)_x) \in \mathscr{G}_x$ for some $s(x) \in \mathscr{F}(V_x)$ where $V_x \subseteq U$ are open subsets containing x.

Now $t_x = \varphi_{V_x}(s(x))_x$. We can make V_x smaller such that $t|_{V_x} = \varphi_{V_x}(s(x)) \in \mathscr{G}(V_x)$. $\varphi(s(x)|_{V_x \cap V_y}) = t|_{V_x \cap V_y} \varphi(s(y)|_{V_x \cap V_y})$. Thus $s(x)|_{V_x \cap V_y} = s(y)|_{V_x \cap V_y}$.

Patch: There exists a unique $s \in \mathscr{F}(U)$ such that $s|_{V_x} = s(x) \in \mathscr{F}(V_x)$ for all $x \in U$. $\varphi_U(s)|_{V_x} = \varphi_{V_x}(s|_{V_x}) = \varphi_{V_x}(s(x)) = t|_{V_x}$. Thus $\varphi_U(s) = t \in \mathscr{G}(U)$. $\square$

Remark: $\mathscr{F}$ is a sheaf on X, $U \subseteq X$ is open. Define $\mathscr{F}|_U$ to be the sheaf on U by $\Gamma(V, \mathscr{F}|_U) = \Gamma(V, \mathscr{F})$

Example: Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Define sheaves $\mathscr{F}, \mathscr{G}$ by $\mathscr{F}(U) = \{f : U \to \mathbb{R} : f \text{ is locally constant}\}$ and $\mathscr{G}(U) = \{g : U \to \mathbb{R} : \frac{\partial g}{\partial \theta} = 1\}$. If $U \subsetneq S^1$ open then $\mathscr{F}|_U \simeq \mathscr{G}_U$ by $f(x, y) \mapsto f(x, y) + \operatorname{Arg}(x, y)$, so $\mathscr{F}_x \simeq \mathscr{F}_x$ for all $x \in S^1$ but $\mathscr{F}$ and $\mathscr{G}$ are not isomorphic as $\mathscr{F}(S^1) = \mathbb{R}$ and $\mathscr{G}(S^1) = \emptyset$.

Sheafification

Let $\mathscr{F}$ be a presheaf on X, define a sheaf $\mathscr{F}^+$ as follows: $U \subseteq X$ open set $\mathscr{F}^+(U)$ to be the set of all functions $s : U \to \coprod_{x \in U} \mathscr{F}_x$ such that

1. $s(x) \in \mathscr{F}_x$ for all x

2. $\forall x \in U$ there exists an open set V with $x \in V \subseteq U$ and $t \in \mathscr{F}(V)$ such that $s(y) = t_y \in \mathscr{F}_y$ forall $y \in V$.

Definition 72. *We define a morphism $\Theta : \mathscr{F} \to \mathscr{F}^+$ by $t \in \mathscr{F}(U)$, $\Theta_U(t) = [x \mapsto t_x] \in \mathscr{F}^+(U)$.*

Exercise: $\Theta_x : \mathscr{F}_x \to \mathscr{F}_x^+$.

Proposition 22. *Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. $\varphi : \mathscr{F} \to \mathscr{G}$ is any morphism. Then there exists a unique $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \varphi^+ \circ \Theta$.*

Proof. Let $s \in \mathscr{F}^+(U)$. i.e. $s : U \to \coprod_{x \in U} \mathscr{F}_x$. For $x \in U$ choose $t(x) \in \mathscr{F}(V_x)$ s.t. $x \in V_x \subset U$ open and $s(y) = t(x)_y$ for all $y \in V_x$.

Set $\tau(x) = \varphi_{V_x} = \varphi_{V_x}(t(x)) \in \mathscr{G}(V_x)$.

If $y \in V_x$, then $\tau(x)_y = \varphi(t(x))_y = \varphi_y(t(x)_y) = \varphi_y(s(y)) \in \mathscr{G}_y$. Thus, $\tau(x_1)|_{V_{x_1} \cap V_{x_2}} = \tau(x_2)|_{V_{x_1} \cap V_{x_2}}$, which gives that $\{\tau(x)\}$ glue to $\tau \in \mathscr{G}(U)$. Set $\varphi_U(s) = t$. Exercise: Check the details! $\square$

Definition 73. *$\varphi : \mathscr{F}' \to \mathscr{F}$ is a morphism of sheaves.*

We say φ is injective if φ_U is injective for all $U \subseteq X$. If φ is injective, then we say $\mathscr{F}' \subseteq \mathscr{F}$ is a subsheaf as $\mathscr{F}'(U) \subseteq \mathscr{F}(U)$.

<u>Exercise</u>: φ injective iff $\varphi_x : \mathscr{F}'_x \to \mathscr{F}_x$ is injective for all $x \in X$.

Consequence: If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves such that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ injective for all $U \subseteq X$ open, then $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective.

Proof. Check: $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ injective for all $p \in X$. If $\varphi_p(s_p) = 0 \in \mathscr{G}_p$ for $s \in \mathscr{F}(U)$, $p \in U$ then $\varphi_U(s)_p = 0$ so $\varphi_U(s)|_V = 0$ for $p \in V \subseteq U$, so $\varphi_V(s|_V) = 0$. Thus $s|_V = 0$, so $s_P = 0$. $\square$

In particular: If a presheaf $\mathscr{F}$ is a subpresheaf of a sheaf $\mathscr{G}$ then $\mathscr{F}^+$ is a subsheaf of $\mathscr{G}$.

Definition 74 (Kernel and Image)**.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups.*

Then $\ker \varphi =$ the sheaf $U \mapsto \ker(\varphi_U) \subseteq \mathscr{F}(U)$.

$\operatorname{Im} \varphi =$ the sheafification of the presheaf $U \mapsto \operatorname{Im}(\varphi_U) \subset \mathscr{G}(U)$.

Note, $\ker \varphi \subset \mathscr{F}$, $\operatorname{Im} \varphi \subset \mathscr{G}$ are subsheaves, and φ injective iff $\ker \varphi = 0$.

Definition 75 (Surjective)**.** *φ is surjective if $\operatorname{Im}(\varphi) = \mathscr{G}$.*

WARNING: φ surjective DOES NOT IMPLY that φ_U is surjective.

Exercise: φ surjective iff $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ surjective for all $p \in X$.

Definition 76 (Quotient Sheaf)**.** *$\mathscr{F}' \subset \mathscr{F}$ a subsheaf, then $\mathscr{F}/\mathscr{F}'$ = the sheafification of $[U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)]$*

We have a surjective morphism $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ which has kernel $\mathscr{F}'$.

Notation: A sequence of sheaves $\to \mathscr{F}^i \xrightarrow{\phi_i} \mathscr{F}^{i+1} \xrightarrow{\phi_{i+1}} \mathscr{F}^{i+2} \to \dots$ is a complex if $\phi_{i+1} \circ \phi_i = 0$ for all i and is exact if $\operatorname{Im} \phi_i = \ker \phi_{i+1}$ for all i.

Equivalently, complexes and exact sequences of the stalks in the category of abelian groups.

Example, $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact iff $\mathscr{F}' \subset \mathscr{F}$ and $\mathscr{F}'' \simeq \mathscr{F}/\mathscr{F}'$.

Definition 77. *$f : X \to Y$ a continuous map, $\mathscr{F}$ a sheaf on X, then $f_*\mathscr{F}$ is a sheaf on Y defined by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$.*

Example: X a variety, $Y \subseteq X$ closed, $i : Y \to X$ the inclusion, $U \subseteq X$ open, then $\mathscr{I}_Y(U) = \{f \in \mathscr{O}_X(U) | f(y) = 0, \forall y \in U \cap Y\}$, $\mathscr{I}_Y \subseteq \mathscr{O}_X$ is a subsheaf of ideals. Then $\mathscr{O}_X(U)/\mathscr{I}_Y(U)$ are the regular functions $U \cap Y \to k$ which can be extended to all of U.

$\mathscr{O}_X/\mathscr{I}_Y \simeq i_*\mathscr{O}_Y$ because we can extend locally. We have exact sequence $0 \to \mathscr{I}_Y \to \mathscr{O}_X \to i_*\mathscr{O}_Y \to 0$, which will often be written $0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \mathscr{O}_Y \to 0$.

Example, $f : X \to y$ a morphism of SWFs, we get morphism $f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$ by $\mathscr{O}_Y(V) \to f_*(\mathscr{O}_X(V)) = \mathscr{O}_X(f^{-1}(V))$, $h \mapsto h \circ f = f^*h$.

Exercise: Find a morphism $f : X \to Y$ of varieties such that $f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism, but f is NOT an isomorphism.

Definition 78 (Inverse Image Sheaf)**.** *Let $f : X \to Y$ continuous, $\mathscr{G}$ a sheaf on Y. $U \subseteq X$ open, define $pre - f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$.*

We have maps $\mathscr{G}(V) \to pre - f^{-1}\mathscr{G}(U)$, $s \mapsto f^{-1}s$, $f(U) \subseteq V$. $pre - f^{-1}\mathscr{G}(U) = \{f^{-1}s | s \in \mathscr{G}(V), V \supseteq f(U)\}$. $f^{-1}s = 0 \iff s|_W = 0$ where $f(U) \subseteq W \subseteq V$. This is a presheaf on X. We define $f^{-1}\mathscr{G}$ to be the sheafification of this presheaf.

Special Case: $X \subseteq Y$ is a subset, $i : X \to Y$ the inclusion, $\mathscr{G}|_X = i^{-1}\mathscr{G}$.

Example: $X \subseteq Y$ open, $pre - i^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq U} \mathscr{G}(V) = \mathscr{U}$. It is already a sheaf, and so the special case just mentioned above holds.

Exercise: $(f^{-1}\mathscr{G})_p = \mathscr{G}_{f(p)}$.

Adjoint Property

$f : X \to Y$ continuous, $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, let $\varphi : \mathscr{G} \to f_*\mathscr{F}$ be a morphism of sheaves on Y. $U \subseteq X$ open, $V \supseteq f(U)$, then $\mathscr{G}(V) \xrightarrow{\varphi_V} \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ and maps are compatible with restrictions of $\mathscr{G}$, so this induces $\psi_U : pre - f^{-1}\mathscr{G}(U) \to \mathscr{F}(U)$, which gives a morphism $\psi : pre - f^{-1}\mathscr{G} \to \mathscr{F}$, we sheafify to get $\psi^+ : f^{-1}\mathscr{G} \to \mathscr{F}$.

Exercise: $\hom(\mathscr{G}, f_*\mathscr{F}) \to \hom(f^{-1}\mathscr{G}, \mathscr{F}) : \varphi \mapsto \psi^+$ is an isomorphism of abelian groups.

Category Theory Interpretation: f^{-1} is a left adjoint functor to f_* and f_* is a right adjoint functor to f^{-1}.

Let X be a variety (ringed-space)

Definition 79 ($\mathscr{O}_X$-module)**.** *An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ on X such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for all open $U \subseteq X$, such that if $V \subseteq U$ open then $\mathscr{F}(U) \to \mathscr{F}(V)$ is $\mathscr{O}_X(U)$-homomorphism (ie, $f \in \mathscr{O}_X(U)$, $m \in \mathscr{F}(U)$, $(f \cdot m)|_V = f|_V \cdot m|_V$.)*

An $\mathscr{O}_X$-homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves such that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is an $\mathscr{O}_X(U)$-homomorphism for all $U \subseteq X$ open.

Note $\ker\varphi \subseteq \mathscr{F}$ and $\operatorname{Im}\varphi \subseteq \mathscr{G}$ are sub $\mathscr{O}_X$-modules, $\mathscr{F}$ is an $\mathscr{O}_X$-module implies that $\mathscr{F}_P$ is an $\mathscr{O}_{X,P}$-module.

Definition 80 (Tensor Product)**.** *$\mathscr{F}, \mathscr{G}$, $\mathscr{O}_X$-modules, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = [U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)]^+$*

Exercise: $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_p = \mathscr{F}_p \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_p$

Definition 81 (Locally Free)**.** *An $\mathscr{O}_X$-module $\mathscr{F}$ is locally free if $\exists \bigcup_\alpha U_\alpha = X$ an open cover such that $\mathscr{F}|_{U_\alpha} \simeq \mathscr{O}_{U_\alpha}^{\otimes r}$.*

Example: $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, let $m \in \mathbb{Z}$.

Definition 82. *A sheaf $\mathscr{O}(m) = \mathscr{O}_{\mathbb{P}^n}(m)$ of $\mathscr{O}_{\mathbb{P}^n}$-modules is $\Gamma(U, \mathscr{O}(m)) = \{h \in k[\pi^{-1}(U)] : h(\lambda x) = \lambda^m h(x) \forall x \in \pi^{-1}(U), \lambda \in k^*\}$.*

Note: $f \in S = k[x_0, \dots, x_n]$ homogeneous of degree > 0. $k[\pi^{-1}(D_+(f))] = k[D(f)] = S_f$. So $\Gamma(D_+(f), \mathscr{O}(m)) = (S_f)_m$.

$\mathscr{O}(m)$ is an invertible $\mathscr{O}_{\mathbb{P}^n}$-module (invertible means locally free of rank 1).

On $U_i = D_+(x_i) : \mathscr{O}_{U_i} \to \mathscr{O}(m)|_{U_i}$, $h \mapsto x_i^m h$. Note that $\mathscr{O}_{\mathbb{P}^n}(0) = \mathscr{O}_{\mathbb{P}^n}$. $\Gamma(\mathbb{P}^n, \mathscr{O}(m)) = S_m$ when $m \geq 0$ and 0 else.

Lemma 14. *$\mathscr{O}(m) \otimes_{\mathbb{P}^n} \mathscr{O}(p) \simeq \mathscr{O}(m+p)$.*

Proof. $U \subseteq \mathbb{P}^n$ open. $\Gamma(U, \mathscr{O}(m)) \otimes \Gamma(U, \mathscr{O}(p)) \to \Gamma(U, \mathscr{O}(m+p))$, $f \otimes g \mapsto fg$. Sheafification gives a map $\mathscr{O}(m) \otimes \mathscr{O}(p) \to \mathscr{O}(m+p)$, restricting to U we have $\mathscr{O}_{U_i} \otimes \mathscr{O}_{U_i} \to \mathscr{O}_{U_i}$. □

Consequence: $\mathscr{O}(m) \simeq \mathscr{O}(p)$ iff $m = p$. If $m \leq p$ and $\mathscr{O}(m) \simeq \mathscr{O}(p)$ implies that $\mathscr{O}(m-p) \simeq \mathscr{O}(p-p)$, on the right we have nonzero global sections, on the left we only do if $m-p$ nonnegative, so $m = p$.

Later: Any invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathscr{O}(m)$, $m \in \mathbb{Z}$.

Coherent Sheaves

Let X be an affine variety and $A = k[X]$, let M be an A-module.

Definition 83 (Quasi-Coherent Sheaf). *$\tilde{M} = [U \mapsto M \otimes_A \mathscr{O}_X(U)]^+$ is called a quasi-coherent $\mathscr{O}_X$-module.*

Note: $M \otimes_A \mathscr{O}_X(D(f)) = M \otimes_A A_f = M_f$.

Examples $\tilde{A} = \mathscr{O}_X$, $Y \subseteq X$ closed, $I = I(Y) \subset A$. $\tilde{I} = \mathscr{I}_Y \subset \mathscr{O}_X$.

Exercise: $\widetilde{A/I} = i_* \mathscr{O}_Y$.

Claim: $\tilde{M}_p = M_{I(p)}$ for $p \in X$, $I(p) = I(\{p\}) \subset A$.

$M_{I(p)} \to \tilde{M}_p$ by $m/f \mapsto (m \otimes 1/f)_p$, this tensor product is in $\tilde{M}(D(F))$. Surjectivity is clear.

Injective: if $m/f \mapsto 0 \in \tilde{M}_p$ then $(m \otimes 1/f)|_{D(h)} = 0$ for some $h \in A \backslash I(p)$, so $m \otimes 1/f = 0 \in M \otimes_A A_h = M_h$, thus $h^n m = 0 \in M$, and so $m/f = 0 \in M_{I(p)}$.

Consequence $\tilde{M}(U)$ =set of functions $s : U \to \coprod_{p \in U} M_{I(p)}$ such that $\forall p \in U$, there exists $p \in V \subset U$ and $m \in M, f \in A$ such tthat $s(p) = m/f \in M_{I(q)}$ for all $q \in V$.

Proposition 23. *$\tilde{M}(D(f)) \simeq M_f$.*

Corollary 17. *1. $\mathscr{O}_X(D(f)) = A_f$*

2. $\Gamma(X, \tilde{M}) = M$.

We now prove the proposition.

Proof. $\psi : M_f \to \tilde{M}(D(f))$ the obvious m

It is injective, as if $\psi(m/f^n) = 0$ then $m/f^n = 0 \in M_{I(p)}$ for all $p \in D(f)$. Thus, for all $p \in D(F)$, there exists $h_p \in A \backslash I(p)$ with $h_p m = 0 \in M$. $D(f) \subseteq$

$\bigcup D(h_p) \Rightarrow V(f) \supseteq V(\{h_p\})$, so $f \in I(V(\{h_p\}))$ means $f^N = \sum_p a_p h_p$ for $a_p \in A$. $f^N m = \sum a_p h_p m = 0$, thus $m/f^n = 0 \in M_f$.

Surjectivity: Let $s \in \tilde{M}(D(f))$, there exists a cover $D(f) = \bigcup_{i=1}^r V_i$, $m_i \in M$, $h_i \in A$ such that $s = m_i/h_i$ on V_i. WLOG $V_i = D(g_i)$ and $V_i = D(h_i)$ (replace $m_i \mapsto g_i m_i$ by $h_i \mapsto g_i h_i$).

On $D(h_i h_j) = D(h_i) \cap D(h_j)$, $s = m_i/h_i = m_j/h_j \in \tilde{M}(D(h_i h_j))$, injectivity for $D(h_i h_j) : m_i/h_i = m_j/h_j \in M_{h_i h_j}$. So $(h_i h_j)^N (h_j m_i - h_i m_j) = 0 \in M$. Replace $m_i \mapsto h_i^N m_i$, $h_i \mapsto h_i^{N+1}$, $h_j m_i = h_i m_j$.

$D(f) \subseteq \bigcup D(h_i)$ so $f^n = \sum a_i h_i$ where $a_i \in A$. Set $m = \sum a_i m_i \in M$. Claim: $s = \psi(m/f^n)$.

For all j, $h_j m = \sum_i h_j a_i m_i = \left(\sum_i a_i h_i\right) m_j = f^n m_j$.

So $m/f^n = m_j/h_j = s$ on $D(h_j)$. □

Definition 84 (Quasi-Coherent and Coherent)**.** *Let X be any variety. An $\mathscr{O}_X$-mdule $\mathscr{F}$ is quasi-coherent if there exists an open affine cover $X = \bigcup U_i$ and $k[U_i]$-modules M_i such that $\mathscr{F}|_{U_i} \simeq \tilde{M}_i$ as $\mathscr{O}_{U_i}$-modules.*

$\mathscr{F}$ is coherent if M_i finitely generated $k[U_i]$-modules for all i.

Examples:

1. All locally free $\mathscr{O}_X$-modules are coherent, $U = \operatorname{Spec} -m(A)$, $\tilde{A^{\otimes r}} = \mathscr{O}_U^{\otimes r}$.

2. $Y \subseteq X$ closed, $\mathscr{I}_Y$ and $i_* \mathscr{O}_Y$ are coherent $\mathscr{O}_X$-modules.

Example: $X = \mathbb{A}^1$, $\mathscr{F}(U) = \begin{cases} \mathscr{O}_{\mathbb{A}^1}(U) & 0 \notin U \\ 0 & 0 \in U \end{cases}$. This is called the extension of the $\mathscr{O}_{\mathbb{A}^1 \setminus \{0\}}$ by zeros. $\mathscr{F}$ is an $\mathscr{O}_X$-module, but is not quasi-coherent, as if $U \subseteq X$ is open affine and $0 \in U$, then $\Gamma(U, \mathscr{F}) = 0$ but $\mathscr{F}|_U$ is not the zero sheaf.

Exercises:

1. $\mathscr{F}$ is quasi-coherent iff $\mathscr{F}|_U \simeq \tilde{\mathscr{F}(U)}$ for all open affine $U \subseteq X$.

2. $\mathscr{F}$ is coherent implies $\mathscr{F}(U)$ is finitely generated $k[U]$-module for all affine open U.

3. $f : X \to Y$ a morphism of varieties.

   (a) f affine implies that $f_* \mathscr{O}_X$ quasi-coherent.

(b) f finite $f_*\mathscr{O}_X$ coherent.

Example: M a finitely generated A-module, $X = \operatorname{Spec}-m(A)$. Then $\tilde{M}$ is locally free $\mathscr{O}_X$-module of rank r iff M is a projective A-module of const. rank r.

($\tilde{M}$ loc free iff $X = \bigcup D(f_i)$, $\tilde{M}_{D(f_i)} \simeq \mathscr{O}_{D(f_i)}^{\otimes} r$ iff $M_{f_i} = A_{f_i}^{\otimes r}$ iff M projective.)

Recall: X is complete implies $\Gamma(X, \mathscr{O}_X) = k$. More general fact: $\mathscr{F}$ a coherent sheaf on a complete variety X then $\dim_k \Gamma(X, \mathscr{F}) < \infty$. We will use this without proof. (Projective case in Hartshorne, General in EGA.)

Note that $\Gamma(\mathbb{A}^1, \mathscr{O}_{\mathbb{A}^1}) = k[t]$ which has infinite dimension.

Pushforward, Pullback

Let $f : X \to Y$ be a morphism of varieties. $\mathscr{F}$ an $\mathscr{O}_X$-module, then $f_*\mathscr{F}$ is a $f_*\mathscr{O}_X$-module, we have ring homomorphism $f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$, and so $f_*\mathscr{F}$ is an $\mathscr{O}_Y$-module.

Let $\mathscr{G}$ be a $\mathscr{O}_Y$-module $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module. $(f^{-1}h \cdot f^{-1}s = f^{-1}(hs))$

$\mathscr{O}_Y \to f_*\mathscr{O}_X$ is the same as $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ by the adjoint property.

Definition 85 (Pullback)**.** *Define $f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, we call it the pullback.*

Examples

1. $f^*\mathscr{O}_Y = \mathscr{O}_X$.
2. $(f^*\mathscr{G})_p = (f^{-1}(G))_p \otimes_{(f^{-1}\mathscr{O}_Y)_p} \mathscr{O}_{X,p} = \mathscr{G}_{f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p}$.
3. $U \subseteq Y$ open, $i : U \to Y$ inclusion, $i^*\mathscr{G} = \mathscr{G}|_U \otimes_{\mathscr{O}_Y|_U} \mathscr{O}_U = \mathscr{G}|_U$.

Adjoint Property

$f : X \to Y$ a morphism of SWFs. $\mathscr{G}$ an $\mathscr{O}_Y$-module. We have $f^{-1}\mathscr{O}_Y$-homomorphism $f^{-1}\mathscr{G} \to f^*\mathscr{G}$ by $s \mapsto s \otimes 1$. This gives an $\alpha : \mathscr{O}_Y$-homomorphism $\mathscr{F} \to f_*f^*\mathscr{G}$.

Lemma 15. *$\mathscr{G}$ is an $\mathscr{O}_Y$-module, $\mathscr{F}$ is an $\mathscr{O}_X$-module. Then* $\hom_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \simeq \hom_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$.

Proof. Given $\psi : f^*\mathscr{G} \to \mathscr{F}$ we obtain $\phi : \mathscr{G} \xrightarrow{\alpha} f_*f^*\mathscr{G} \xrightarrow{f_*\psi} f_*\mathscr{F}$.

Given $\phi : \mathscr{G} \to f_*\mathscr{F}$, we obtain $\tilde{\phi} : f^{-1}\mathscr{G} \to \mathscr{F}$ which is an $f^{-1}\mathscr{O}_Y$-hom. Take $\psi : f^*\mathscr{G} = f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \to \mathscr{F}$ by $s \otimes h \mapsto h \cdot \tilde{\phi}(s)$. □

Functoriality $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphism of SWFs. If $\mathscr{F}$ is a sheaf on X, $g_*(f_*\mathscr{F}) = (gf)_*\mathscr{F}$.

Proposition 24. *1. $\mathscr{G}$ on Z implies that $(gf)^{-1}\mathscr{G} = f^{-1}(g^{-1}\mathscr{G})$*

2. $\mathscr{G}$ an $\mathscr{O}_Z$-module implies that $(gf)^*\mathscr{G} = f^*(g^*\mathscr{G})$.

Proof. We will prove case 2.

id : $(gf)^*\mathscr{G} \to (gf)^*\mathscr{G}$ gives $\mathscr{G} \to (gf)_*(gf)^*\mathscr{G} = g_*f_*(gf)^*\mathscr{G}$ gives $g^*\mathscr{G} \to f_*(gf)^*\mathscr{G}$ which gives $f^*(g^*\mathscr{G}) \to (gf)^*\mathscr{G}$.

We have a global homomorphism, so enough to check stalks. $f^*(g^*\mathscr{G})_p = (g^*\mathscr{G})_{f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p}$. This is $(\mathscr{G}_{g(f(p))} \otimes_{\mathscr{O}_{Z,gf(p)}} \mathscr{O}_{Y,f(p)}) \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p} = \mathscr{G}_{gf(p)} \otimes_{\mathscr{O}_{Z,gf(p)}} \mathscr{O}_{X,p} = ((gf)^*\mathscr{G})_p$ □

Let $f : X \to Y$ be a morphism of SWFs. $\mathscr{G}$ is an $\mathscr{O}_Y$-module, and $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module.

Definition 86 (Pullback)**.** *$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is an $\mathscr{O}_X$-module.*

$f^* : \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces a map $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Some sections: $\sigma \in \mathscr{G}(V)$, we set $f^*\sigma = f^{-1}\sigma \otimes 1 \in \Gamma(f^{-1}(V), f^*\mathscr{G})$. The stalks $(f^*\mathscr{G})_p = \mathscr{G}_{f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p}$.

If $Z \xrightarrow{g} X \xrightarrow{f} Y$, then $(fg)^*\mathscr{G} = g^*(f^*\mathscr{G})$.

Corollary 18. *If $\mathscr{G}$ is a locally free $\mathscr{O}_Y$-module, then $f^*\mathscr{G}$ is a locally free $\mathscr{O}_X$-module of the same rank.*

Proof. Let $Y = \bigcup V_i$ be an open cover such that $\mathscr{G}|_{V_i} \simeq \mathscr{O}_{V_i}^{\oplus r}$. Set $U_i = f^{-1}(V_i) \subset X$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U_i \arrow[u, "p"] \arrow[r, "f'"] & V_i \arrow[u, "q"'] \end{tikzcd}$$

$f^*\mathscr{G}|_{U_i} = p^*f^*\mathscr{G} = f'^*q^*\mathscr{G} \simeq f'^*q^*(\mathscr{O}_{V_i}^{\oplus r}) = \mathscr{O}_{U_i}^{\oplus r}$ □

Lemma 16. *$f : X \to Y$ a morphism, $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ is a short exact sequence of $\mathscr{O}_Y$-modules. Then $f^*\mathscr{G}' \to f^*\mathscr{G} \to f^*\mathscr{G}'' \to 0$ is an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{G}''$ is locally free, then the first map is injective.*

Proof. On stalks we start with $0 \to \mathscr{G}'_{f(p)} \to \mathscr{G}_{f(p)} \to \mathscr{G}''_{f(p)} \to 0$ exact. Tensor produce is right exact and gets us to f^*, ad so we have the first part of the theorem immediately.

If $\mathscr{G}''$ is locally free, then $\mathscr{G}''_{f(p)}$ is a free $\mathscr{O}_{Y,f(p)}$-module, and so the original sequence is split-exact. □

Definition 87 (Generated by Finitely Many Global Sections)**.** *The $\mathscr{O}_X$-module $\mathscr{F}$ is generated by finitely many global sections iff $\exists$ a surjective map $\mathscr{O}_X^{\oplus m} \to \mathscr{F}$.*

Equivalently, $\exists s_1, \dots, s_m \in \Gamma(X, \mathscr{F})$ such that $\mathscr{F}_p$ generated by $(s_1)_p, \dots, (s_m)_p$ as an $\mathscr{O}_{X,p}$-module.

Example: Any quasi-coherent $\mathscr{O}_X$-module, if X is affine (this is just generated by global sections, requires coherent to be generated by finitely many)

Example: $\mathscr{O}_{\mathbb{P}^n}(1)$ is generated by $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathscr{O}(1))$.

Suppose that $f : X \to \mathbb{P}^n$ is a morphism, then $\mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathscr{O}(1) \to 0$ exact implies that $\mathscr{O}_X^{\oplus n+1} \to f^*\mathscr{O}(1) \to 0$ is exact, so $f^*\mathscr{O}(1)$ is generated by global sections $f^*(x_0), \dots, f^*(x_n)$.

Proposition 25. *X a variety, $\mathscr{L}$ invertible $\mathscr{O}_X$-module generated by global sections $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$. Then $\exists! f : X \to \mathbb{P}^n$ such that $f^*\mathscr{O}(1) \simeq \mathscr{L}$ and $f^*(x_i) \leftrightarrow s_i$.*

Proof. Set $U_i = \{p \in X | (s_i)_p \notin \mathfrak{m}_p \mathscr{L}_p\}$.

U_i open: If $V \subseteq X$ open with $\mathscr{L}|_V \simeq \mathscr{O}_V$, then $\mathscr{L}|_V$ is generated by $t \in \Gamma(V, \mathscr{L})$ so we write $s_i = h_i t$ on V with $h_i \in k[V]$. Then $U_i \cap V = \{p \in V | h_i(p) \neq 0\}$ is open in V.

Note: $\mathscr{L}$ is generated by $s_0, \dots, s_n$ implies that $X = \bigcup_{i=1}^n U_i$, and $\mathscr{O}_{U_i} \simeq \mathscr{L}|_{U_i}$ implies that $1 \mapsto s_i|_{U_i}$.

On U_i, we can write $s_i = h_{ij} s_j$ for some $h_{ij} \in k[U_i]$. We define a map $g : U_i \to \mathbb{P}^n$ by $f(p) = (h_{i0}(p) : \dots : h_{in}(p))$.

The maps are compatible: On $U_i \cap U_j$, $h_{\ell j} s_\ell = s_j = h_{ij} s_i = h_{ij} h_{\ell i} s_\ell$, so $h_{\ell j} = h_{\ell i} h_{ij}$. The map on $U_\ell : p \mapsto (h_{\ell 0}(p) : \dots : h_{\ell n}(p)) = h_{\ell i}(p) h_{i0}(p) : \dots : h_{\ell i}(p) h_{in}(p)$. Thus, we have a morphism $f : X \to \mathbb{P}^n$.

Claim: $\exists$ isomorphism $\mathscr{L} \to f^*\mathscr{O}(1)$ by $s_i \mapsto f^*(x_i)$. On U_i, we define $\mathscr{L}|_{U_i} \to f^*\mathscr{O}(1)|_{U_i}$ by $hs_i \mapsto hf^*(x_i)$. This means that $s_\ell = h_{i\ell} s_i \mapsto h_{i\ell} f^*(x_i)$, we must check that $f^*(x_\ell) = h_{i\ell} f^*(x_i)$. The definition of f implies that $(x_\ell / x_i) \circ f = \frac{h_{i\ell}}{h_{ii}} = h_{i\ell}$, so $f^*(x_\ell) = f^*(\frac{x_\ell}{x_i} x_i) = f^*(\frac{x_\ell}{x_i}) f^*(x_i) = h_{i\ell} f^*(x_i)$.

And now we prove uniqueness: If $f : X \to \mathbb{P}^n$ is any morphism such that $\mathscr{L} \simeq f^*\mathscr{O}(1)$ and $s_i \leftrightarrow f^*(x_i)$ on U_i, $h_{i\ell}s_i = f^*(x_\ell) = f^*(\frac{x_\ell}{x_i}x_i) = f^*(\frac{x_\ell}{x_i})f^*(x_i) = f^*(\frac{x_\ell}{x_i})s_i$, so $f^*(x_\ell/x_i) = h_{i\ell}$ on U_i. □

Definition 88 (Very Ample Sheaf)**.** *Let $\mathscr{L}$ be an invertible sheaf on X. $\mathscr{L}$ is very ample iff $\mathscr{L}$ is generated by (finitely many) global sections and the map $f : X \to \mathbb{P}^n$ given by generators $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$ is an isomorphism $f : X \to W \subseteq \mathbb{P}^n$ locally closed.*

Exercise: $\mathscr{O}_{\mathbb{P}^n}(m)$ is very ample iff $m \geq 1$.

Definition 89 (PGL)**.** *$PGL(n) = GL(n+1)/k^*$.*

Exercise: $PGL(n)$ is an affine algebraic group.

FACT: Every invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathscr{O}(m)$ for some m.

Corollary 19. *$\mathrm{Aut}(\mathbb{P}^n) \simeq PGL(n)$.*

Proof. $PGL(n) \leq \mathrm{Aut}(\mathbb{P}^n)$ is trivial.

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be an automorphism. The fact implies that $f^*\mathscr{O}(1) \simeq \mathscr{O}(m)$ for some m. In fact, $\binom{n+m}{m} = \dim_k \Gamma(\mathbb{P}^n, \mathscr{O}(m)) = \dim \Gamma(f^*\mathscr{O}(1)) = \dim \Gamma(\mathscr{O}(1)) = n+1$, so $m = 1$.

$f^*x_0, \dots, f^*x_n \in \Gamma(\mathbb{P}^n, \mathscr{O}(1))$ form a basis. We write $f^*(x_i) = \sum_{i=0}^n a_{ij}x_j$ for $a_{ij} \in k$. Then $A = (a_{ij}) \in GL(n+1)$.

Define $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ by $\varphi(x_0 : \dots : x_n) = (\sum a_{0j}x_j : \dots : a_{nj}x_j)$. $\varphi^*(\frac{x_\ell}{x_i}) = \frac{\sum a_{\ell j}x_j}{\sum a_{ij}x_j} = f^*(\frac{x_\ell}{x_i})$.

Thus, $f = \varphi \in PGL(n)$. □

Corollary 20. $\mathbb{P}^n$ *is not an algbriac group.*

Chapter 9

Normal Varieties

Definition 90 (Normal Variety)**.** *X is irreducible. Then X is normal iff $\mathscr{O}_{X,P}$ is normal (integrally closed) for all p.*

Example: Nonsingular varieties.
Note: X affine, then X is normal iff $k[X]_{\mathfrak{m}}$ normal for all maximal $\mathfrak{m}$ iff $k[X]$ is normal.
Exercise: If A is a domain, $S \subseteq A$ multiplicative, then $\overline{S^{-1}A} = S^{-1}\overline{A}$.

Definition 91 (Normalization)**.** *If f is an affine variety, $k[X] \subset k(X)$, then $\overline{k[X]} \subseteq k(X)$ is the integral closure. The normalization of X is $\overline{X} =$ $\operatorname{Spec} -m(\overline{k[X]})$.*

As we have the inclusion $k[X] \to \overline{k[X]}$, we get a projection map $\overline{X} \to X$ which is finite.

Lemma 17. *$\varphi : U \to X$ morphism of affines, φ an open embedding iff $\exists f_1, \dots f-n \in k[X]$ such that $(\varphi^* f_1, \dots, \varphi^* f_n) = (1) \subset k[U]$ and $\varphi^* : k[X]_{f_i} \to k[U]_{\varphi^* f_i}$ is an isomorphism for all i.*

Proof. $\Rightarrow$: Take open cover $U = \bigcup_{i=1}^r D(f_i)$, $f_i \in k[X]$.
$\Leftarrow$: Set $V = \bigcup_{i=1}^r D(f_i) \subseteq X$. $(\varphi^* f_1, \dots, \varphi^* f_r) = (1) \subseteq k[U]$ implies that $\varphi(U) \subseteq V$. Thus, we have that $\varphi : \varphi^{-1}(D(f_i)) \to D(f_i)$ is an isomorphism, so $\varphi : U \to V$ isomorphism. □

Lemma 18. *Assume X affine, $U \subseteq X$ is an open affine, then $\bar{U} \subseteq \bar{X}$ open affine.*

Proof. $k[X] \subseteq k[U] \subseteq k(X)$. Thus, $\overline{k[X]} \subseteq \overline{k[U]}$, so we have a morphism $\varphi : \bar{U} \to \bar{X}$. Take $f_1, \ldots, f_n \in k[X]$ as in lemma wrt $U \subset X$.

$(f_1, \ldots, f_n) = (1) \subseteq \overline{k[U]}$, and $k[\bar{U}]_{f_i} = \overline{k[U]_{f_i}} = \overline{k[X]_{f_i}} = k[\bar{X}]_{f_i}$. And so, the lemma implies that φ is an open embedding. □

Exercise: Given pre-varieties $X_1, \ldots, X_n$, open subsets $U_{ij} \subseteq X_i$ and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ such that $U_{ii} = X_i$, $\varphi_{ii} = \mathrm{id}$, for all i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi ij$ on $U_{ij} \cap U_{ik}$, then $\exists!$ prevariety X with morphisms $\psi_i : X_i \to X$ such that $\psi_i : X_i \to \text{open}\subset X$ is an isomorphism, $X = \bigcup_{i=1}^n \psi_i(X_i)$, $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

Definition 92 (Normal)**.** *A variety X is normal iff X is irreducible and $\mathscr{O}_{X,P}$ are normal for all p.*

Construction: X an irreducible variety, $X = X_1 \cup \ldots \cup X_n$ open affine cover, set $U_{ij} = X_i \cap X_j$. Then U_{ij} is affine, have $\psi_{ij} : U_{ij} \to U_{ji}$ nbe the identity. Now $\bar{U}_{ij} \subseteq \bar{X}_i$ and still have $\phi_{ij} : \bar{U}_{ij} \to \bar{U}_{ji}$ is the identity, so the satisfy the hypotheses of the exercise. Thus, there exists a prevariety $\bar{X} = \bar{X}_1 \cup \ldots \cup \bar{X}_n$. We call this the normalization of X.

Note: $\overline{k[X_i]}$ is a finitely generated $k[X_i]$-module, so we have finite $\pi : \bar{X}_i \to X_i$, which we can glue to a morphism $\pi : \bar{X} \to X$.

Exercise: $\pi : \bar{X} \to X$ is finite. (Check that $\pi^{-1}(X_i) = \bar{X}_i$)

Exercise: $\varphi : X \to Y$ affine morphism of pre-varieities. Then Y is separated implies that X is serpated, and so $\bar{X}$ is an irreducible normal separated variety.

Example: X irred curve implies $\pi : \bar{X} \to X$ resoluton of singularities.

Definition 93 (Local Ring along Subvariety)**.** *Let X be a variety, $V \subseteq X$ irreducible and closed. Then $\mathscr{O}_{X,V} = \varinjlim_{U \subseteq X, U \cap V \neq} \mathscr{O}_X(U)$.*

If X is irreducible, then $\mathscr{O}_{X,V} = \{f \in k(X) : f$ is defined at one point of $V\}$.

General Case: $U \subseteq X$ open affine, $U \cap V \neq \emptyset$, $P = I(U \cap V) \subseteq k[U]$, $\mathscr{O}_{X,V} = k[U]_P$.

Definition 94 (Regular along V)**.** *X is regular along V if $\mathscr{O}_{X,V}$ is a regular local ring. i.e., the maximal ideal in $\mathscr{O}_{X_V}$ is generated by* $\dim(\mathscr{O}_{X,V}) = \operatorname{codim}(V; X)$ *elements. This happens iff $V \not\subseteq X_{sing}$.*

Chapter 10

Divisors

Let X be a normal variety.

Definition 95 (Prime Divisor)**.** *A prime divisor on X is a closed irreducible subvariety of codimension 1.*

Note: Y a prime divisor implies that $\mathscr{O}_{X,Y}$ is a normal Nötherian local ring of dimension 1. That is, $\mathscr{O}_{X,Y}$ is a DVR.
Consequences

1. $\operatorname{codim}(X_{sing}; X) \geq 2$.

2. Have valuation map $v_Y : k(X)^* \to \mathbb{Z}$ for each prime divisor $Y \subseteq X$.

Lemma 19. *Let $f \in k(X)^*$. Then $v_Y(f) = 0$ for all but finitely many Y.*

Proof. Show $v_Y(f) < 0$ for finitely many Y. Set $U \subseteq X$ open set where f defined. $Z = X \setminus U$. $v_Y(f) < 0$ iff $f \notin \mathscr{O}_{X,Y}$ iff f is not defined at any point of Y iff $Y \subseteq Z$ component. $\square$

Definition 96 (Divisor Group)**.** *Define* $\operatorname{Div}(X)$ =*free abelian group generated by all prime divisors.*

An element $D = \sum n_i[Y_i]$ is a finite sum and is called a Weil Divisor.

Definition 97. *For $f \in k(X)^*$, set $(f) = \sum_Y v_Y(f) \cdot [Y] \in \operatorname{Div}(X)$.*

Note $(f^{-1}) = -(f)$, $(fg) = (f) + (g)$ Thus $k(X)^* \to \operatorname{Div}(X)$ by $f \mapsto (f)$ is a group homomorphism.

Definition 98 (Class Group)**.** *Define* $\mathrm{C}\ell(X) = \mathrm{Div}(X)/\{(f) : f \in k(X)^*\}$.

Example: $\mathrm{C}\ell(\mathbb{A}^n) = 0$, as every hypersurface corresponds to a prime divisor.

Remark: X complex, $\dim(X) = n$, then $\mathrm{C}\ell(X) \leftrightarrow H_{2n-2}(X;\mathbb{Z})$.

Remark: X irreducible but not normal, we can still define $\mathrm{C}\ell(X)$, use $v_Y(f/g) = length_{\mathscr{O}_{X,Y}}(\mathscr{O}_{X,Y}/(f)) - length_{\mathscr{O}_{X,Y}}(\mathscr{O}_{X,Y}/(g))$.

Divisors on $\mathbb{P}^n$

Note: All prime divisors are hypersurfaces $Y = V_+(h)$ where $h \in S = k[x_0, \dots, x_n]$ is an irreducible form.

Definition 99. *Degree of a Divisor* $\deg : \mathrm{Div}(\mathbb{P}^n) \to \mathbb{Z}$ *by* $\deg(\sum m_i[Y_i]) = \sum m_i \deg Y_i$.

Let $f \in k(\mathbb{P}^n)^*$, $m_i \in \mathbb{Z}$, $g = \prod_{i=1}^r h_i^{m_i}$, $h_i \in S$ irreducible form. Then $\sum m_i \deg(h_i) = 0$, so $Y_i = V_+(h_i) \subseteq \mathbb{P}^n$ is a prime divisor, so $v_{Y_i}(h_i) = 1$, $v_{Y_i}(f) = m_i$.

$(f) = \sum_{i=1}^r m_i[Y_i]$ implies that $\deg(f) = \sum m_i \deg(h_i) = 0$, thus, $\deg : \mathrm{C}\ell(\mathbb{P}^n) \to \mathbb{Z}$ is well-defined.

Claim: Isomorphism.

Surjective: $H \subseteq \mathbb{P}^n$ hyperplane, $\deg(m[H]) = m$. Injective: Let $D = \sum m_i[Y_i] \in \mathrm{Div}(\mathbb{P}^n)$, suppose $\deg(D) = 0$, then $Y_i = V_+(h_i)$, $h_i \in S$ irred form, so $\sum m_i \deg(h_i) = \deg(D) = 0$, so $f = \prod h_i^{m_i} \in k(\mathbb{P}^n)^*$ and $D = (f)$.

Later: X nonsingular implies $\mathrm{C}\ell(X) \simeq \mathrm{Pic}(X)$, thus $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$.

X normal, $Y \subseteq X$ prime divisor implies that $\mathscr{O}_{X,Y}$ is a DVR.

Theorem 34. *R normal Nötherian domain implies $R = \cap_{\mathrm{ht}\,\mathfrak{p}=1} R_{\mathfrak{p}}$ where* $\mathrm{ht}\,\mathfrak{p} = \dim R_{\mathfrak{p}}$*, the max m such that $\exists 0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_m = \mathfrak{p}$.*

Corollary 21. *If X is normal, $f \in k(X)^*$, then $f \in k[X]$ iff $v_Y(f) \geq 0$ for all $Y \subseteq X$ prime divisors.*

Lemma 20. *R Nötherian, then R is a UFD iff all prime ideals of height one are principal.*

Proof. $\Rightarrow$: Assume R a UFD. $P \subseteq R$ prime of height 1, let $x \in P$ be an irreducible element, $0 \subsetneq (x) \subseteq P$, so $P = (x)$.

$\Leftarrow$: Exercise: R any Nötherian domain then every element of R is a product of irreducible elements.

Unique Factorization: Show $x \in R$ irred and $x|fg$ implies that $x|f$ or $x|g$, ie, $(x) \subseteq R$ is prime, let $P \supseteq (x)$ min prime, PIT implies $\mathrm{ht}(P) = 1$ implies $P = (y)$, $x = ay$, $a \in R$ a unit. $\square$

Proposition 26. *X irreducible affine variety, $k[X]$ a UFD iff X normal and $\mathrm{C}\ell(X) = 0$.*

Proof. $\Rightarrow$: UFD implies normal. Let $Y \subseteq X$ a prime divisor, $P = I(Y) \subseteq k[X]$ prime of height 1. $P = (h) \subseteq k[X]$, $h \in k[X]$. So $(h) = [Y]$ implies $[Y] = 0 \in \mathrm{C}\ell(X)$.

$\Leftarrow$: Let $P \subseteq k[X]$ prime of height 1, $Y = V(P) \subseteq X$ a prime divisor, $[Y] = 0 \in \mathrm{C}\ell(X)$ so $[Y] = (h) \in \mathrm{Div}(X)$, $h \in k(X)^*$. $v_Z(h) \geq 0$ for all $Z \subseteq X$ prime divisors implies $h \in k[X]$. Claim: $P = (h) \subseteq k[X]$. $\supseteq$ is clear. Let $g \in P$, then $v_Y(g) \geq 1$, so $v_Z(g/f) \geq 0$ for all Z, so $g/f \in k[X]$, and $g = af \in (f)$. $\square$

Proposition 27. *X normal, $Z \subseteq X$ is a proper closed subset, $U = X \setminus Z$. Then*

1. *$\mathrm{C}\ell(X) \to \mathrm{C}\ell(U)$ by $[Y] \mapsto [Y \cap U]$ if $Y \cap U \neq \emptyset$ and 0 else is surjective*
2. *If $\mathrm{codim}(Z; X) \geq 2$, then $\mathrm{C}\ell(X) = \mathrm{C}\ell(U)$.*
3. *If Z prime divisor, then $\mathbb{Z} \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(U) \to 0$ is exact.*

Proof.

1. Well defined $\mathrm{Div}(X) \to \mathrm{Div}(U)$. $f \in k(X)^*$, $(f) \mapsto (f|_U)$ (because if $Y \subseteq X$ is a prime divisor, $Y \cap U \neq \emptyset$ then $\mathscr{O}_{X,Y} = \mathscr{O}_{U,U\cap Y}$).

   Thus, $\mathrm{C}\ell(X) \to \mathrm{C}\ell(U)$ is well defined. Surjective: If $V \subseteq U$ a prime divisor, $\bar{V} \subseteq X$ is a prime divisor $[\bar{V}] \mapsto [V]$.
2. $\mathrm{Div}(X) = \mathrm{Div}(U)$, so $(f) = (f|_U)$.
3. If $D = \sum n_Y[Y] \mapsto 0 \in \mathrm{C}\ell(U)$, then $\exists f \in k(X)^*$: $v_Y(f) = n_Y$ for all $Y \neq Z$. $D - (f) = m[Z] \Rightarrow D = m[Z] \in \mathrm{C}\ell(X)$.

$\square$

Example: $X = V(xy - z^2) \subset \mathbb{A}^3$.

Exercise: X (above) is normal. $L = V(y) \cap X = V(y, z)$ is a prime divisor on X.

Max ideal of $\mathscr{O}_{X,L}$ is generated by z, $y = \frac{z^2}{x} \in \mathscr{O}_{X,L}$.

Set $U = X \setminus L$ affine, $k[U] = (k[x,y,z]/(xy - z^2))_y = k[y,z]_y$ is a UFD, so $\mathrm{C}\ell(U) = 0$. $\mathbb{Z} \to \mathrm{C}\ell(X) \to \mathrm{C}\ell(U) = 0$. So $\mathrm{C}\ell(X) = \{m[L] : m \in \mathbb{Z}\}$, $y \in k(X)^*$. $(y) = v_L(y)[L] = v_L(z^2)[L] = 2[L]$. Thus, $\mathrm{C}\ell(X) = \mathbb{Z}/2\mathbb{Z}$ or $\mathrm{C}\ell(X) = 0$. As $k[x,y,z]/(xy - z^2)$ is not a UFD, $\mathrm{C}\ell(X) = \mathbb{Z}/2\mathbb{Z}$.

Picard Group

Invertible $\mathscr{O}_X$-module = line bundle.

Let X be any variety, $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles, then $\mathscr{L}_1 \otimes_{\mathscr{O}_X} \mathscr{L}_2$ is a line bundle. $\mathscr{L}$ is invertible implies that we can define $\mathscr{L}^{-1} = [U \mapsto \hom_{\mathscr{O}_U}(\mathscr{L}|_U, \mathscr{O}_U)]$.

Exercise: $\mathscr{L}^{-1}$ is an invertible $\mathscr{O}_X$-module and $\mathscr{L} \otimes \mathscr{L}^{-1} \simeq \mathscr{O}_X$.

Definition 100 (Picard Group)**.** $\mathrm{Pic}(X) = \{$*isomorphism classes of invertible sheaves on* $X\}$.

This is a group under tensor product.

Notation: X irreducible variety, $\mathscr{L}$ an invertible sheaf on X and $s \in \mathscr{L}(U)$, $t \in \mathscr{L}(V)$ are nonzero sections. Take $W \subset U \cap V$ open such that $\mathscr{L}|_W \simeq \mathscr{O}_W$ generated by $u \in \mathscr{L}(W)$. Then $s|_W = fu$ and $t|_W = gu$, $f, g \in k[W]$. Define $s/t = f/g \in k(X)^*$.

Example: $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$. Define $f : X \dashrightarrow \mathbb{A}^n \subset \mathbb{P}^n$. $f(x) = (s_1/s_0(x), \dots, s_n/s_0(x)) = (s_1/s_0(x) : \dots : s_n/s_0(x) : 1)$. If $s_0, \dots, s_n$ generate $\mathscr{L}$, then f extends to a morphism $f : X \to \mathbb{P}^n$.

X is a normal variety, $s \in \mathscr{L}(U)$, $s \neq 0$, $Y \subseteq X$ is a prime divisor, take $V \subseteq X$ open such that $\mathscr{L}|_V \simeq \mathscr{O}_V$ generated by $t \in \mathscr{L}(V)$ and $V \cap Y \neq \emptyset$.

Definition 101. $V_Y(s) = V_Y(s/t)$.

Well defined, if $V' \subseteq X$, $V' \cap Y \neq \emptyset$, $\mathscr{L}|_{V'}$ generated by $y' \in \mathscr{L}(V')$ implies that t/t' nowhere vanishing function on $V \cap V'$, so t/t' is a unit in $\mathscr{O}_{X,Y}$.

Thus, $V_Y(s/t') = V_Y(s/t \cdot t/t') = V_Y(s/t) + 0$.

Definition 102. $(s) = \sum_Y V_Y(s)[Y] \in \mathrm{Div}(X)$.

Note: If $s' \in \mathscr{L}(U')$ then $(s') = (s) + (s'/s)$. Then $(s') = (s) \in \mathrm{C}\ell(X)$, so all nonzero sections of a line bundle are equivalent in the class group.

Thus, we have a well defined map $\mathrm{Pic}(X) \to \mathrm{C}\ell(X)$ by $\mathscr{L} \mapsto (s)$.

Check: $s_1 \in \mathscr{L}_1(U)$, $s_2 \in \mathscr{L}_2(U)$, then $s_1 \otimes s_2 \in \mathscr{L}_1 \otimes \mathscr{L}_2(U)$ and $(s_1 \otimes s_2) = (s_1) + (s_2)$, so this map is a group homomorphism.

Cartier Divisors

X normal.

Definition 103 (Cartier Divisor)**.** *A Cartier is a Weil Divisor* $D = \sum n_i[Y_i]$ *which is locally principal. I.E. there exists an open covering* $\bigcup_{i=1}^n U_i = X$ *such that* $D|_{U_i} = 0 \in \mathrm{C}\ell(U_i)$ *for all* i.

Note: $D|_{U_j} = (f_j)$ where $f_j \in k(U_j)^*$. We can think about D as the collection $\{f_j\}$ of these generators.

Definition 104 (Cartier Class Group)**.** $\mathrm{CaC}\ell(X) = \{$*Cartier Divisors on* $X\}/\{(f) : f \in k(X)^*\}$.

Recall: $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module, $s \in \mathscr{L}(U)$ nonzero section, then $(s) = \sum_{Y \texttt{ prime}} v_Y(s) \cdot [Y] \in \mathrm{Div}(X)$ where $v_Y(s) = v_Y(s/t)$ for $t \in \mathscr{L}(V)$ generator of $\mathscr{L}|_V \simeq \mathscr{O}_V$, $Y \cap V \neq \emptyset$.

Note: (s) is Cartier.

Thus, we have a group homomorphism $\mathrm{Pic}(X) \to \mathrm{CaC}\ell(X) \subset \mathrm{C}\ell(X)$.

<u>Line Bundles from Divisors</u>

Let $D = \sum n_Y[Y] \in \mathrm{Div}(X)$.

Definition 105. *$\mathscr{O}_X$-module $\mathscr{L}(D)$ or $\mathscr{O}_X(D)$* $\Gamma(U, \mathscr{L}(D)) = \{f \in k(X)^* | v_Y(f) \geq -n_Y$ *for all prime divisors Y such that* $Y \cap U \neq \emptyset\} \cup \{0\}$.

Example: $\mathscr{L}(0) = \mathscr{O}_X(0) = \mathscr{O}_X$.

Example: $X = \mathbb{P}^1$, $Q = (a : b) \in X$, $D = n[Q]$, $Q = V_+(h)$, $h = bx_0 - ax_1 \in k[x_0, x_1]$. So $\mathscr{O}_{\mathbb{P}^1}(n[Q]) \simeq \mathscr{O}_{\mathbb{P}^1}(n)$ by $f \mapsto h^n f$.

Note: If $h \in k(X)^*$ then $\mathscr{O}_X(D + (h)) \to \mathscr{O}_X(D)$ by $f \mapsto hf$ is an isomorphism. $v_Y(f) \geq -n_Y - v_Y(h) \iff v_Y(fh) \geq -n_Y$.

Consequence: D is a Cartier Divisor implies that $\mathscr{O}_X(D)$ is an invertible $\mathscr{O}_X$-module.

If $D|_U = (h) \in \mathrm{Div}(U)$, then $\mathscr{O}_X(D)|_U = \mathscr{O}_U((h)) \simeq \mathscr{O}_U$. Note, this says we have a map $\mathrm{CaC}\ell(X) \to \mathrm{Pic}(X)$ by $D \mapsto \mathscr{O}_X(D)$.

WARNING: If $f \in \Gamma(U, \mathscr{O}_X(D))$ then f a rational function. The notation (f) means two things!

Proposition 28. $\mathrm{Pic}(X) \simeq \mathrm{CaC}\ell(X)$ *as abstract groups.*

Proof. Will check that $\mathrm{Pic}(X) \overset{\leftarrow}{\to} \mathrm{CaC}\ell(X)$ are inverse maps.

Let $D = \sum n_Y[Y]$ a Cartier Divisor. Set $V = X \setminus (\cup_{n_Y < 0} Y) \subseteq X$ open. Then $1 \in k(X)^*$ is a section of $\mathscr{O}_X(D)$ over V. $(v_Y(1) \geq -n_Y \iff n_Y \geq 0)$.

Claim: $(1) = D \in \mathrm{Div}(X)$. If $D|_U = (g) \in \mathrm{Div}(U)$, then $\mathscr{O}_X(D)|_U \simeq \mathscr{O}_U$ gneerated by $h^{-1} \in \Gamma(U, \mathscr{O}_X(D))$, $Y \cap U \neq \emptyset$ implies that $v_Y(1) = v_Y(1/h^{-1}) = v_Y(h)$. Thus, $(1)|_U = (h)|_U = D|_U$.

Thus, $\mathrm{CaC}\ell(X) \to \mathrm{Pic}(X) \to \mathrm{CaC}\ell(X)$ is the identity.

Let $\mathscr{L}$ be a line bundle on X. $t \in \Gamma(U, \mathscr{L})$ a non-zero section.

Note: If $0 \neq s \in \mathscr{L}(V)$, then $Y \cap V \neq \emptyset$ implies $v_Y(s/t) = v_Y(s) - v_Y(t) \geq -v_Y(t)$, and so $s/t \in \Gamma(V, \mathscr{O}_X((t)))$.

Claim: $\mathscr{L} \simeq \mathscr{O}_X((t))$ by $s \mapsto s/t$. If $\mathscr{L}|_V \simeq \mathscr{O}_V$ gneerated by $u \in \mathscr{L}(V)$ then $(t)|_V = (t/u) \in \mathrm{Div}(V)$ implies that $\mathscr{O}_X((t))|_V$ is generated by u/t as $u \mapsto u/t$ we get $\mathscr{L}|_V \simeq \mathscr{O}_X((t))|_V$. □

Examples:

1. $\mathrm{Pic}(\mathbb{A}^n) = \mathrm{CaC}\ell(\mathbb{A}^n) \subset \mathrm{C}\ell(\mathbb{A}^n) = 0$, so all line bundles on $\mathbb{A}^n$ are trivial.

   Fact: Any locally free $\mathscr{O}_{\mathbb{A}^n}$-module of finite rank is trivial.

2. $\mathbb{P}^n = \bigcup_{i=0}^n D_+(x_i)$, $\mathrm{C}\ell(D_+(x_i)) = 0$, so all Weil divisors are Cartier, thus $\mathrm{Pic}(\mathbb{P}^n) = \mathrm{CaC}\ell(\mathbb{P}^n) = \mathrm{C}\ell(\mathbb{P}^n) = \mathbb{Z}$.

   By the maps we have, any line bundle is isomorphic to $\mathscr{O}_{\mathbb{P}^n}(m[H])$, where $H \subset \mathbb{P}^n$ is a hyperplane. $I(H) = (h)$, $\mathscr{O}_{\mathbb{P}^n}(m[H]) \simeq \mathscr{O}_{\mathbb{P}^n}(m)$ by $f \mapsto h^m f$. Thus, $\mathrm{Pic}(\mathbb{P}^n) = \{\mathscr{O}(m)\}$.

3. $X = V(xy - z^2) \subset \mathbb{A}^3$. $L = V(y) \cap X$, $I(L) = (y, z) \subset k[\mathbb{A}^3]$. Claim: $[L]$ is not Cartier. Otherwise there exists open affine $U \subset X$ such that $P = (0,0,0) \in U$ with $[L \cap U] = (f|_U) \in \mathrm{Div}(U)$. Thus $f \in k[U]$ and $I(L \cap U) = (f) \subset k[U]$, so $I(L) \cdot \mathscr{O}_{X,P} = (y, z) \subset \mathscr{O}_{X,P}$ is principal.

   But $P \in X$ is a singular point, so $\dim_k(\mathfrak{m}_P/(m)_P^2) = 3$, so $\{x, y, z\}$ is a basis, and so $\dim((y,z) + \mathfrak{m}_P^2/\mathfrak{m}_P^2) = 2$, so $(y, z) \subset \mathscr{O}_{X,P}$ is not principal, which is a contradiction. Thus $\mathrm{CaC}\ell(X) = \mathrm{Pic}(X) = 0 \neq \mathrm{C}\ell(X)$.

Note: $\mathscr{O}_{X,P}$ is not a UFD, as $(y, z) \subset \mathscr{O}_{X,P}$ is height 1 prime but not principal.

Definition 106 (Locally Factorial)**.** *An irreducible variety X is locally factorial if $\mathscr{O}_{X,P}$ is a UFD for all $p \in X$.*

Example: Nonsingular implies locally factorial implies normal.

Proposition 29. *X locally factorial implies* $\mathrm{Pic}(X) = \mathrm{C}\ell(X)$.

Proof. Show that any prime divisor $[Y]$ is Cartier. First: $U = X \setminus Y$, $[Y]|_U = 0$. Let $P \in Y$, then $I(Y) \cdot \mathscr{O}_{X,P} \subset \mathscr{O}_{X,P}$ is a height 1 prime, so $I(Y) \cdot \mathscr{O}_{X,P} = (f) \subset \mathscr{O}_{X,P}$, $f \in \mathscr{O}_{X,P} \subset k(X)$.

Note: $v_Y(f) = 1$, if $Z \neq Y$ prime divisor, $p \in Z$, then $f \in \mathscr{O}_{X,Z}$ (defined at P) and $f \notin I(Z) \cdot \mathscr{O}_{X,P}$. Thus, $v_Z(f) = 0$, and we have $(f) = [Y] + \sum n_i[Z_i]$ where $p \notin Z_i$ for all i.

Set $U = X \setminus (\cup Z_i)$ open in X, $p \in U$. Then $[Y]|_U = (f)|_U \in \mathrm{Div}(U)$ principal, so $[Y]$ is Cartier. $\square$

Example: $X = V(xy - z^2) \subset \mathbb{A}^3$, $X_0 = X \setminus \{0, 0, 0\}$, X_0 is nonsingular, so $\mathrm{Pic}(X_0) = \mathrm{C}\ell(X_0) = \mathrm{C}\ell(X) = \mathbb{Z}/2\mathbb{Z}$, so there exists a unique nontrivial line bundle on X_0 which is NOT equal to the restriction of a line bundle on X.

Definition 107 (Affine, Finite)**.** *Let $f : X \to Y$ be a morphism of varieties. f is affine if $f^{-1}(V) \subset X$ is affine for all $V \subset Y$ is open affine.*

f is finite if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module.

Exercise: Enough that this is true for an open affine cover of Y.

Examples: X, Y affine, $f : X \to Y$ morphism is affine.

$X \subset Y$ closed, then the inclusion is finite.

10.1 Divisors on Non-Singular Curves

Recall that X nonsing complete curve implies that X is projective. $X \subset C_K$ open, $K = k(X)$, $X = C_K$.

Lemma 21. *Let X be a complete, nonsingular curve, then any nonconstant morphism $f : X \to Y$ is finite.*

Proof. WLOG: Y is a curve. Thus, $f^* : k(Y) \subset k(X)$ is a finite field extension. Take $V \subseteq Y$ open affine, $k[V] \subset k(Y)$. Set $A = \overline{k[V]} \subset k(X)$, then A is a finitely generated $k[V]$-module.

$U = \mathrm{Spec}-m(A)$ a nonsingular curve, $k(U) = k(X) \Rightarrow$ we have diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\subset"] & Y \arrow[d, "\subset"] \\ U \arrow[r] & V \end{tikzcd}$$

Claim: $f^{-1}(V) = U$. $x \in f^{-1}(V) \Rightarrow k[V] \subseteq \mathscr{O}_{X,x}$, so $A \subset \mathscr{O}_{X,x}$, thus $\mathscr{O}_{X,x} = A_P$ for some $P \subset A$ prime.

Thus, $x = P \in U = \mathrm{Spec}-m(A)$. $\square$

Definition 108 (Degree of f)**.** *Let $f : X \to Y$ be a finite, dominant morphism, then* $\deg(f) = [k(X) : k(Y)]$.

Pullback of Divisors on Curves

$f : X \to Y$ a finite morphism of nonsingular curves. $Q \in Y$, $\mathfrak{m}_Q = (t) \subseteq \mathscr{O}_{Y,Q}$. If $f(P) = Q$ then $f^* : \mathscr{O}_{Y,Q} \to \mathscr{O}_{X,P}$, $f^*t \in \mathfrak{m}_P$.

Definition 109. $f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X) : [Q] \to \sum_{P \in f^{-1}(Q)} v_P(t)[P]$.

Alternatively, if $D \in \mathrm{Div}(Y)$, set $V = Y - \mathrm{Supp}(D)$, then $s = 1 \in \Gamma(V, \mathscr{L}(D))$. Note: $(s) = D$. Then $f^*s \in \Gamma(f^{-1}(V), f^*\mathscr{L}(D))$ is the pullback.

Exercise: $f^*D = (f^*s) \in \mathrm{Div}(X)$.

Definition 110 (Torsion Free)**.** *Let R be a domain, M an R-module. M is torsion free if $\forall a \in R, x \in M$, then $ax = 0$ implies $a = 0$ or $x = 0$.*

Fact: Any f.g. torsion-free module over a PID is free.

Definition 111 (Degree of a Divisor)**.** *X a nonsingular curve, $D = \sum n_i[P_i] = \sum n_i P_i \in \mathrm{Div}(X)$. Set* $\deg(D) = \sum n_i$

Warning: If X is not complete, then deg is not defined on $\mathrm{C}\ell(X)$.

Proposition 30. *$f : X \to Y$ is a finite morphism of nonsingular curves, $D \in \mathrm{Div}(Y)$. Then* $\deg(f^*D) = \deg(f)\deg(D)$.

Proof. ETS if $Q \in Y$ a point, then $\deg(f^*Q) = \deg(f)$. $V \subseteq Y$ open affine with $Q \in V$. Then $f^{-1}(V) = \mathrm{Spec}\,{-m}(A) \subset X$, $A = \overline{k[V]} \subset k(X)$.

$Q \subset k[V]$ a max ideal, set $B = A_Q = (k[V] \setminus Q)^{-1}A$. A finitely generated $k[V]$-module implies B f.g. $k[V]_Q = \mathscr{O}_{Y,Q}$-module. $\mathscr{O}_{Y,Q}$ a DVR, B torsion free, so B is free $\mathscr{O}_{Y,Q}$-module.

$\mathrm{rank}_{\mathscr{O}_{Y,Q}}(B) = \dim_{k(Y)} k(X) = \deg(f)$. $\mathfrak{m}_Q = (t) \subset \mathscr{O}_{Y,Q}$. $\mathscr{O}_{Y,Q}/t\mathscr{O}_{Y,Q} = k$. Thus, $\dim_k(B/tB) = \deg(f)$.

Note: points in $f^{-1}(Q)$ correspond to max ideals in $P \subset A$ such that $P \cap k[V] = Q$, which correspond to max ideals in $A_Q = B$.

Write $f^{-1}(Q) = \{P_1, \ldots, P_s\}$, $P_i \subseteq A$ max ideals, $B = \cap_{i=1}^s B_{P_i} \Rightarrow tB = \cap_{i=1}^s tB_{P_i} = \cap_{i=1}^s (tB_{P_i} \cap B)$.

By the Chinese Remainder Theorem, $B/tB \simeq \oplus_{i=1}^s B/(tB_{P_i} \cap B)$.

Injective: clear

Surjective: $t \in P_i$ for all i, B_{P_i} DVR, so $tB_{P_i} = (P_iB_{P_i})^{n_i}$, so $tB_{P_i} \cap B \subseteq P_iB$ and $tB_{P_i} \cap B \not\subseteq P_jB$ for $j \neq i$.

Thus, this is an isomorphism after $\otimes B_{P_i}$ (only the i^{th} summand survives).

Now: $B/(tB_{P_i} \cap B) = (B/(tB_{P_i} \cap B))_{P_i} = (B/tB)_{P_i} = B_{P_i}/tB_{P_i} = \mathscr{O}_{X,P_i}/(t)$. Thus $\dim_k B/(tB_{P_i} \cap B) = v_{P_i}(t)$.

Thus, $\deg(f^*Q) = \sum v_{P_i}(t) = \dim_k(B/tB) = \deg(f)$. $\square$

Lemma 22. *$h \in k(Y)^*$ implies $f^*((h)) = (f^*h)$. $f^*h = h \circ f \in k(X)$.*

Proof. Let $P \in X$, $Q = f(P) \in Y$. $\mathfrak{m}_P = (s) \subseteq \mathscr{O}_{X,P}$. $\mathfrak{m}_Q = (t) \subset \mathscr{O}_{Y,Q}$.

$h = ut^m$, $u \in \mathscr{O}_{Y,Q}$ a unit. $f^*t = vs^n$, $v \in \mathscr{O}_{X,P}$ a unit.

Coef of $[P]$ in $f^*((h)) = v_Q(h)v_P(t) = mn$.

$f^*h = f^*(ut^m) = (f^*u)(f^*t)^m = (f^*u)v^ms^{nm}$, so coef of $[P]$ in (f^*h) is nm. $\square$

So, we have a group homomorphism $f^* : \mathrm{C}\ell(Y) \to \mathrm{C}\ell(X)$.

Corollary 22. *X complete nonsing curve, $f \in k(X)^*$ implies $\deg((f)) = 0$.*

Proof. f is defined on open $U \subset X$. Then $f : U \to \mathbb{A}^1 \subset \mathbb{P}^1$ is a regular function. As $\mathbb{P}^1$ is complete, f extends to $f : X \to \mathbb{P}^1$. As X is complete, f is finite.

$k[\mathbb{A}^1] = k[t]$, $f^*(t) = f \in k(X)$. $(f) = (f^*t) = f^*((t))$, so $\deg((f)) = \deg(f^*((t))) = \deg(f)\deg(t)$.

$(t) = [0] - [\infty] \in \mathrm{Div}(\mathbb{P}^1)$ so $\deg((t)) = 0$. $\square$

So if X is a complete nonsingular curve, there exists $\deg : \mathrm{C}\ell(X) \to \mathbb{Z}$.

Notation: $D, D' \in \mathrm{Div}(X)$, $D \sim D'$ iff $D = D' \in \mathrm{C}\ell(X)$.

Proposition 31. *If X complete nonsingular curve, then X rational iff $\exists P \neq Q \in X$ such that $P \sim Q$.*

Proof. $\Rightarrow$: $X = \mathbb{P}^1$, then $P \sim Q$ for all $P, Q \in \mathbb{P}^1$.

$\Leftarrow$: $\exists f \in k(X)^*$ such that $(f) = [P] - [Q] \in \mathrm{Div}(X)$. $f : X \to \mathbb{P}^1$ a morphism. Then $(f) = (f^*t) = f^*((t)) = f^*([0] - [\infty])$.

This tells us that $f^*([0]) = [P]$ and $f^*([\infty]) = [Q]$. So $\deg(f^*[0]) = 1 = \deg(f) \cdot 1$, so f is degree 1, so it is birational. Thus isomorphism. $\square$

Definition 112 (Elliptic Curve)**.** *An elliptic curve is a nonsingular closed plane curve $E \subset \mathbb{P}^2$ such that* $\deg(E) = 3$.

Example: $V_+(zy^2 - x^3 + z^2x) \subset \mathbb{P}^2$.

Claim: No elliptic curve is rational.

Exercise: Set $\mathscr{O}_E(1) = \mathscr{O}_{\mathbb{P}^2}(1)|_E$. Then $\Gamma(E, \mathscr{O}_E(1)) = (S/I(E))_1$, $S = k[x,y,z]$.

Therefore, $\dim_k \Gamma(E, \mathscr{O}_E(1)) = 3$.

Let $L \subset \mathbb{P}^2$ be a line. $L.E = \sum_{P' \in L \cap E} I(L \cdot E; P')[P'] = P + Q + R \in \mathrm{Div}(E)$.

Exercise: $\mathscr{O}_E(1) = \mathscr{O}_{\mathbb{P}^2}([L])|_E = \mathscr{O}_E([L.E])$, so $\exists D (= L.E)$, $D \in \mathrm{Div}(E)$ such that $\deg(D) = 3$ and $\dim_k \Gamma(E, \mathscr{O}_E(D)) = 3$.

Let $D \in \mathrm{Div}(\mathbb{P}^1)$ such that $\deg(D) = 3$. Then $\mathscr{O}_{\mathbb{P}^1}(D) = \mathscr{O}_{\mathbb{P}^1}(3)$, this gives us that $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(D)) = \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(3)) = k[x_0, x_1]_3$. The dimension of this is 4.

Conclude: E is not rational.

Let $X \subset \mathbb{P}^2$ be any nonsingular curve. $\deg : \mathrm{C}\ell(X) \to \mathbb{Z}, [P] \mapsto 1$.

Definition 113. $\mathrm{C}\ell^0(C) = \ker(\deg)$. *So we have a short exact sequence* $0 \to \mathrm{C}\ell^0(X) \to \mathrm{C}\ell(X) \to \mathbb{Z} \to 0$ *that splits, so* $\mathrm{C}\ell(X) = \mathrm{C}\ell^0(X) \oplus \mathbb{Z}$.

Fact: $\mathrm{C}\ell^0$ corresponds to a nonginular complete abelian algebraic group, the Jacobi Variety of X.

Let $L = V_+(f)$, $M = V_+(g)$ be lines in $\mathbb{P}^2$.

$X.L = \sum_P I(X \cdot L; P)P = P_1 + \ldots + P_n$ where $n = \deg(X)$. $X.M = Q_1 + \ldots + Q_n$.

Exercise: $f/g \in k(X)^*$ and $(f/g) = X.L - X.M \in \mathrm{Div}(X)$.

$X.L - X.M = P_1 + P_2 + P_3 - Q_1 - Q_2 - Q_3 = 0 \in \mathrm{C}\ell(X)$ for $X = E$.

Theorem 35. *Let* $P_0 \in E$ *be any point, then* $E \to \mathrm{C}\ell^0(E)$ *by* $P \mapsto P - P_0$ *is bijective.*

Proof. Injective: If $P - P_0 = Q - P_0 \in \mathrm{C}\ell^0(E)$ then $P \sim Q$ so $P = Q$ as E is not rational.

Surjective: Let $M \subset \mathbb{P}^2$ be tangent line to E at P_0. $M.E = 2P_0 + R$, $R \in E$. Let $D \in \mathrm{C}\ell^0(E)$. Write $D = \sum n_i(Q_i - P_0)$ for $Q_i \in E$, $n_i \in \mathbb{Z}$.

Assume that $n_i < 0$. Then L =line through Q_i and R, $L.E = Q_i + R + Q_i'$, $0 = L.E - M.E = Q_i + R + Q_i' - 2P_0 - R$ so $Q_i - P_0 = -(Q_i' - P_0) \in \mathrm{C}\ell^0(E)$.

Replace Q_i by Q_i', $n_i \mapsto -n_i$, WLOG, $n_i \geq 0$.

Claim: $D = P - P_0 \in \mathrm{C}\ell^0(E)$, $P \in E$. Induction on $\sum n_i$.

If $\sum n_i \geq 2$, then $Q_1 - P_0$, $Q_2 - P_0$ have positive coefficients. L =line through Q_1, Q_2, $L.E = Q_1 + Q_2 + Q' \in \mathrm{Div}(E)$.

Let L' be the line through Q' and P_0. Then $L'.E = Q' + P_0 + Q''$. $L.E - L'.E = Q_1 + Q_2 - P_0 - Q'' = 0$, so $(Q_1 - P_0) + (Q_2 - P_0) = (Q'' - P_0) \in \mathrm{C}\ell^0(E)$. $\square$

Example: $\operatorname{char} k \neq 2$, $\lambda \in k$, $\lambda \neq 0, 1$. $E_\lambda = V_+(zy^2 - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$. Take $P_0 = (0:1:0) \in E$ E_λ corresponds to $\mathrm{C}\ell^0(E_\lambda)$ by $P \mapsto P - P_0$. Let $\oplus$ be a group op on W. $Q_1 \oplus Q_2 = Q''$ (Picture omitted)

Fact: Any elliptic curve is isomorphic to E_λ by $P_0 \leftrightarrow (0:1:0)$.

Theorem 36. *E is an algebraic group.*

Proof. ($\operatorname{char} k \neq 2$): WLOG, $E = E_\lambda$, $P_0 = (0:1:0)$. Define $\varphi : E \times E \to E$ by $\varphi(P, Q) = R$ the unique point such that $\exists$ a line L with $L.E = P + Q + R$. It is enough to show that $\varphi : E \times E \to E$ is a morphism.

$P \oplus Q = \varphi(P_0, \varphi(P, Q))$, $-P = \varphi(P_1, P_0)$. Set $U_1 = D_+(z)$ and $U_2 = D_+(y)$ subsets of E. $E = U_1 \cup U_2$.

Show that $(U_i \times U_j) \cap \varphi^{-1}(U_\ell) \to U_\ell$ by φ is a morphism for all $i, j, \ell \in \{0, 1\}$.

$U_1 = V(y^2 - x(x-1)(x-\lambda)) \subset \mathbb{A}^2$. $(U_1 \times U_1) \cap \varphi^{-1}(U_1) \to U_1 \to k$ is the regular function given by

$$(x_1, y_1) \times (x_2, y_2) \quad \mapsto$$
$$\begin{cases} \frac{(y_2-y_1)^2}{(x_2-x_1)^2} - (x_1 + x_2) + 1 + \lambda & \text{if } x_1 \neq x_2 \\ \left(\frac{x_1^2 + x_1 x_2 + x_2^2 + \lambda - (1+\lambda)(x_1+x_2)}{y_1+y_2}\right)^2 + 1 + \lambda - (x_1 + x_2) & \text{if } y_1 + y_2 \neq 0 \end{cases}$$

$\square$

Chapter 11

Differentials

R is a ring, S is a commutative R-algebra, M an S-module.

Definition 114 (R-derivation)**.** *A function $D : S \to M$ is an R-derivation if $D(fg) = fD(g) + gD(f)$ for all $f, g \in S$, $D(f+g) = D(f) + D(g)$, and $D(f) = 0$ for all $f \in R$.*

Remark: the third conditioon holds iff D is an homomorphism of R-modules.

$\Rightarrow$: $f \in R \Rightarrow D(fg) = fD(g)$ and $\Leftarrow$ is an exercise (use $D(1) = 0$).

Definition 115 (Module of Kähler differentials)**.** *F=free S-module with basis $\{d(f)|f \in S\} = \oplus_{f\in S} S \cdot d(f)$. F' =submodule generated by $d(f)$ for $f \in R$, $d(fg) - fd(g) - gd(f)$, $d(f+g) - d(f) - d(g)$.*

We define $\Omega_{S/R} = F/F'$ is the module of Kähler differentials of S over R

We define $d = d_S = d_{S/R} : S \to \Omega_{S/R}$ by $f \mapsto d(f) + F'$. This is the universal R-derivation of S.

It has the universal property that given any R-derivation $D : S \to M$, there exists a unique map S-homomorphism $\tilde{D} : \Omega_{S/R} \to M$ such that $D = \tilde{D} \circ d_S$.

<u>Exercise</u>: Let $P(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ and $f_1, \dots, f_n \in S$, and $D : S \to M$ is an R-derivation. Then $D(P(f_1, \dots, f_n)) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(f_1, \dots, f_n) D(f_i)$.

Consequence: If S generated by $f_1, \dots, f_n$ as an R-algebra, then $\Omega_{S/R}$ is gererated by $d_S(f_1), \dots, d_S(f_n)$ as an S-module.

Proposition 32. *$S = R[x_1, \dots, x_n]$. Then $\Omega_{S/R}$ is the free S-module on $dx_1, \dots, dx_n$.*

Proof. Have a surjectve S-hom from $S^n \to \Omega_{S/R}$ which sends $e_i \mapsto dx_i$. This is surjective. We define $D : S \to S^n$ by $P(x_1, \dots, x_n) \mapsto \left(\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_n}\right)$. By the universal property, there is a unique S-homomorphism $\tilde{D} : \Omega_{S/R} \to S^n$, by definition, $d(x_i) \mapsto D(x_i) = e_i$, so this is an inverse. $\square$

Proposition 33. *Given ring homomorphisms $R \to S \to Y$ then we have an exact sequence of T-modules $\Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0$.*

Proof. $S \to T \xrightarrow{d_T} \Omega_{T/R}$ is an R-deriv of S. So we get S-hom $\varphi : \Omega_{S/R} \to \Omega_{T/R}$ via $\varphi(d_S(f)) = d_T(f)$. Thus, we have a T-hom $\Omega_{S/R} \otimes T \overset{\tilde{\varphi}\to}{\Omega}_{T/R}$ by $\omega \otimes h \mapsto h\varphi(\omega)$.

Note: Image($\tilde{\varphi}$) =submodule of $\Omega_{T/R}$ generated by $d_T(f)$ for $f \in S$. Thus $\Omega_{T/R}/\operatorname{Im}(\tilde{\varphi}) = \Omega_{T/S}$. $\square$

Note: $I \subset S$ an ideal, $T = S/I$, then I/I^2 is a T-module, $T \times I/I^2 \to I/I^2$ by $(f + I) \cdot (h + I^2) = fh + I^2 \in I/I^2$.

Proposition 34. *$T = S/I$. We have an exact sequence of T-modules $I/I^2 \to \Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to 0$, where the first map is given by $h + I^2 \mapsto d_s(h) \otimes 1$.*

Proof. Set M equal to the image of I/I^2 in $\Omega_{S/R} \otimes_S T$. Then M is generated by $\{d_s(h) \otimes 1 | h \in I\}$.

We define $D : T \to (\Omega_{S/R} \otimes T)/M$ by $D(f + I) = (d_S(f) \otimes 1) + M$. This is an R-derivation.

Thus, there is a unique T-hom $\tilde{D} : \Omega_{T/R} \to (\Omega_{S/R} \otimes_S T)/M$ by $d_T(f+I) \mapsto (d_s(f) \otimes 1) + M$ $\square$

Example: $S = R[x_1, \dots, x_n]$, $I = (f_1, \dots, f_p) \subset S$. $T = S/I$. $\Omega_{S/R} \otimes_S T = \oplus_{i=1}^n T dx_i = T^{\oplus n}$.

The image of f_i under $I/I^2 \to T^{\oplus n}$: $d_s(f_i) \otimes 1 = \left(\frac{\partial \bar{f_i}}{\partial x_1}, \dots, \frac{\partial \bar{f_i}}{\partial x_n}\right)$. Set $J = \left(\frac{\partial \bar{f_i}}{\partial x_j}\right) \in Mat(p \times n; T)$ is the Jacobi matrix.

So the image of f_i is $e_i J$, so $\Omega_{T/R} = \operatorname{coker}(I/I^2 \to \Omega_{S/R} \otimes T) = \operatorname{coker}(T^p \xrightarrow{J} T^n)$.

e.g. $T = k[x, y]/(y^2 - x^3 + x)$, so $J = [1 - 3x^2, 2y]$, so $\Omega_{T/k} = T \oplus T/\langle(1 - 3x^2)e_1 + (2y)e_2\rangle$.

Proposition 35. *S an R-algebra, $U \subseteq S$ multiplicatively closed subset, then $\Omega_{U^{-1}S/R} = U^{-1}\Omega_{S/R}$*

Proof. $S \to U^{-1}S \to \Omega_{U^{-1}S/R}$ is an R-derivation. Thus, it induces an S-homomorphism $\Omega_{S/R} \to \Omega_{U^{-1}S/R}$, $d_S(f) \mapsto d(f)$, where d is the universal derivation of $U^{-1}S$.

This induces $U^{-1}S$-hom $U^{-1}\Omega_{S/R} \to \Omega_{U^{-1}S/R}$ by $d_s(f)/u \mapsto u^{-1}d(f)$.

We define $D : U^{-1}S \to U^{-1}\Omega_{S/R}$ by $D(s/u) \mapsto \frac{ud(s)-sd(u)}{u^2}$. Exercise: D is well defined R-derivation.

This induces $\tilde{D} : \Omega_{U^{-1}S/R} \to U^{-1}\Omega_{S/R}$ is the inverse map. □

Let X be a topological space. $\mathscr{R}, \mathscr{S}$ sheaves of rings on X, $\mathscr{R} \to \mathscr{S}$ a ring hom.

Definition 116. *$pre-\Omega_{\mathscr{S}/\mathscr{R}}(U) = \Omega_{\mathscr{S}(U)/\mathscr{T}(U)}$ for $U \subseteq X$ open. For $V \subset U$ open, $\mathscr{S}(U) \to \mathscr{S}(V) \xrightarrow{d} pre - \Omega(V)$ is an $R(U)$-derivation. So, we get $\mathscr{S}(U)$-hom $pre - \Omega(U) \to pre - \Omega(V)$.*

We define $\Omega_{\mathscr{S}/\mathscr{R}} = (pre - \Omega_{\mathscr{S}/\mathscr{R}})^+$, the sheafification.

Let $\varphi : X \to Y$ morphism of varieties, then we have ring hom $\varphi^* : \varphi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Definition 117 (Relative cotangent sheaf)**.** *$\Omega_{X/Y} = \Omega_{\mathscr{O}_X/\varphi^{-1}\mathscr{O}_Y}$ is called the relative cotangent sheaf*

Special case: $X \to \{pt\}$, $\Omega_X = \Omega_{X/k} = \Omega_{X/\{pt\}}$. This is called the cotangent sheaf.

Proposition 36. *$\varphi : X \to Y$ a morphism of affine varieties, then $\Omega_{X/Y} = \widetilde{\Omega_{k[X]/k[Y]}}$*

Proof next time.

As a consequence, $\Omega_{X/Y}$ is always coherent.

Lemma 23. *If $(A, \mathfrak{m})$ is a local Nötherian domain, N a finitely generated A-module, then we set $r = \dim_{A/\mathfrak{m}}(N/\mathfrak{m}N)$. If $r \leq \dim_{A_0}(N_0)$, then N is free of rank r.*

Proof. Nakayama's Lemma implies that N can be generated by r elements. Thus, there exists an exact sequence $0 \to K \to A^r \to N \to 0$, localization is exact, so $0 \to K_0 \to A_0^r \to N_0 \to 0$ is exact, so the last morphism is an isomorphism of vector spaces, so $A_0^r \simeq N_0$, so $K_0 = 0$. Thus, $K = 0$ as it is torsion free and localizes to zero, so $A^r \simeq N$. □

Recall: Let $X \subset \mathbb{A}^n$ be a closed irreducible variety. Let $I = I(X) = (f_1, \ldots, f_s)$. Let $P \in X$. Set $M = I(\{p\}) \subset k[\mathbb{A}^n]$, $M/M^2 \simeq k^n$ via $h + M^2 \mapsto \left(\frac{\partial h}{\partial x_1}(P), \ldots, \frac{\partial h}{\partial x_n}(P)\right)$

So we have an exact sequence $(I + M^2)/M^2 \to M/M^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to 0$. $\mathfrak{m}_P \subset \mathscr{O}_{X,P}$ a max ideal, therefore $k^s \to k^n \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to 0$ is also exact where the first map is $J(P)$, and we call the second ϕ.

If $h \in M$, then $\phi\left(\frac{\partial h}{\partial x_1}(P), \ldots, \frac{\partial h}{\partial x_n}(P)\right) = h + \mathfrak{m}_P^2$.

Note: $\operatorname{rank}(k(X)^s \xrightarrow{J} k(X)^n) \leq c = \operatorname{codim}(X; \mathbb{A}^n)$. (If h is any $(c+1) \times (c+1)$-minor of J, then $h \in k[X]$, and $h(P) = (c+1) \times (c+1)$-minor. $J(P) = 0$ for all $P \in X$. So $h = 0 \in k[X]$.)

Theorem 37. *Assume X is an irreducible variety of dimension r, let $P \in X$. Then P is a nonsingular point iff $\Omega_{X,P} \simeq \mathscr{O}_{X,P}^{\oplus r}$. If $\mathfrak{m}_P$ is generated by $h_1, \ldots, h_r \in \mathfrak{m}_P$, then $dh_1, \ldots, dh_r \in \Omega_{X,P}$ is a basis for $\Omega_{X,P}$.*

Proof. WLOG: $X \subseteq \mathbb{A}^n$ affine. $I(X) = (f_1, \ldots, f_s)$, $J = \left(\frac{\partial f_i}{\partial x_j}\right)$.

$k[X]^s \xrightarrow{J} k[X]^n \to \Omega_{k[X]/k} \to 0$ yields $\mathscr{O}_{X,P}^s \xrightarrow{J} \mathscr{O}_{X,P}^n \to \Omega_{X,P} \to 0$, which we will call $(*)$.

We mod out by $\mathfrak{m}_P$, and get $k^s \xrightarrow{J(P)} k^n \to \Omega_{X,P}/\mathfrak{m}_P\Omega_{X,P} \to 0$.

Thus, $\mathfrak{m}_P/\mathfrak{m}_P^2 \simeq \Omega_{X,P}/\mathfrak{m}_P\Omega_{X,P}$ by $h + \mathfrak{m}_P^2 \mapsto \sum_{j=1}^n \frac{\partial h}{\partial x_j}(P) dx_j$.

Assume that $\Omega_{X,P}$ is free of rank r, then $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = r$, thus P is a nonsingular point.

Assume that $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = r$. $(*) \Rightarrow k(X)^s \xrightarrow{J} k(X)^n \to (\Omega_{X,P})_0 \to 0$ is exact.

Note: $r = \dim_k(\Omega_{X,P}/\mathfrak{m}_P\Omega_{X,P}) \leq \dim_{k(X)}((\Omega_{X,P})_0)$.

The lemma implies that $\Omega_{X,P} \simeq \mathscr{O}_{X,P}^{\oplus r}$. $\square$

Lemma 24. *$\varphi : X \to Y$ is a morphism of affine varieties. Then $\Gamma(X, pre-\Omega_{X/Y}) = \Omega_{k[X]/k[Y]}$*

Proof. $S = \Gamma(X, \varphi^{-1}\mathscr{O}_Y)$, ring homomorphisms $k[Y] \to S \to k[X]$. Thus, $\Omega_{S/k[Y]} \otimes_S k[X] \to \Omega_{k[X]/k[Y]} \to \Omega_{k[X]/S} \to 0$ where the last is $\Gamma(X, pre-\Omega_{X/Y})$, so enough to show that the first map is zero.

Let $f \in \operatorname{Im}(S \to k[X])$. We must show that $df = 0 \in \Omega_{k[X]/k[Y]}$. There exists open cover $X = \cup_{i=1}^n U_i$ such that $f|_{U_i} \in$ image of $\Gamma(U_i, pre-\varphi^{-1}\mathscr{O}_Y) = \varinjlim_{V \supset \varphi(U_i)} \mathscr{O}_Y(V)$

WLOG, $U_i = X_{g_i}$ where $g_i \in k[X]$. Enough to show that $df = 0$ in $\left(\Omega_{k[X]/k[Y]}\right)_{g_i}$ for each i, since $g_i^N df = 0 \in \Omega_{k[X]/k[Y]}$ and $(g_i^N, \ldots, g_n^N) = (1) = k[X]$.

But $\left(\Omega_{k[X]/k[Y]}\right)_{g_i} = \Omega_{k[U_i]/k[Y]}$. So we replace X with U_i, we may assume that $f \in$ image of $\Gamma(X, pre - \varphi^{-1}\mathscr{O}_Y) = \varinjlim_{V \supset \varphi(X)} \mathscr{O}_Y(V)$. I.E. there exists $V \subset Y$ open, $f' \in \mathscr{O}_Y(V)$ such that $\varphi(X) \subset V$ and $f = \varphi^*(f') \in k[X]$. Now $V = \cup_{i=1}^m Y_{h_i}$, $h_i \in k[Y]$, $f' \in k[Y]_{h_i}$, so $h_i^N f' \in k[Y]$ for all i, so $h_i^{N+1} df = d(h_i^{N+1} f) = 0 \in \Omega_{k[X]/k[Y]}$.

Now, $X = \cup X_{h_i} \Rightarrow (h_1^N, \ldots, h_m^N) = (1) \subset k[X]$, so $df = 0$. □

Proposition 37. *$\varphi : X \to Y$ morphism of affines. Then $\Omega_{X/Y} = \widetilde{\Omega_{k[X]/k[Y]}}$*

Proof. Set $\Omega = \Omega_{k[X]/k[Y]}$. We have $\Omega \simeq \Gamma(X, pre - \Omega_{X/Y}) \to \Gamma(X, \Omega_{X/Y})$, this gives an $\mathscr{O}_X$-homomorphism $\tilde{\Omega} \to \Omega_{X/Y}$.

Let $f \in k[X]$. $\Gamma(X_f, \tilde{\Omega}) = \Omega_f = \Omega_{k[X_f]/k[Y]} = \Gamma(X_f, pre - \Omega_{X/Y})$.

$\{X_f\}$ is a basis for the top, and so they have the same stalks. □

Corollary 23. *$\varphi : X \to Y$ any morphism of varieties. Then $\Omega_{X/Y}$ is coherent.*

Proof. Let $Y = \cup V_i$ open affine cover. $\varphi^{-1}(V_i) = \cup U_{ij} \subseteq X$ is an open affine cover of X. $\Omega_{X/Y}|_{U_{ij}} = \tilde{\Omega}_{k[U_{ij}]/k[V_i]}$. □

Corollary 24. *If X irreducible, then X is nonsingular iff Ω_X is a locally free $\mathscr{O}_X$-module.*

Example: $X = \mathbb{P}^1$, $\Omega_{\mathbb{P}^1}$ is a line bundle. The projective coordinate ring is $k[x_0, x_1]$. Set $t = \frac{x_1}{x_0} \in k(\mathbb{P}^1)^*$. $t \in \mathscr{O}_{\mathbb{P}^1}(D_+(x_0))$, $dt \in \Gamma(D_+(x_0), \Omega_{\mathbb{P}^1})$. Find $(dt) \in \mathrm{Div}(\mathbb{P}^1)$.

$U_i = D_+(x_i)$, $U_0 = \mathbb{A}^1 \subset \mathbb{P}^1$. If $p \in U_0$, then $t - p$ generated $\mathfrak{m}_p$, so $\Omega_{\mathbb{P}^1,p}$ is generated by $d(t-p) = dt$, so $v_p(dt) = 0$ for all $p \in U_0$.

$k[U_1] = k[s]$, $s = t^{-1}$, $dt = d(s^{-1}) = -s^{-2} ds$, $v_\infty(dt) = v_\infty(s^{-2}) = -2$. Thus $(dt) = -2[\infty] \in \mathrm{Div}(\mathbb{P}^1)$, and so $\Omega_{\mathbb{P}^1} \simeq \mathscr{O}_{\mathbb{P}^1}(-2[pt]) = \mathscr{O}_{\mathbb{P}^1}(-2)$.

Example: $E = E_\lambda \subset \mathbb{P}^2$ an elliptic curve. Then $\Omega_E \simeq \mathscr{O}_E$.

Linear Systems

Let X be a nonsingular projective variety. $D = \sum n_Y[Y] \in \mathrm{Div}(X)$. We say that D is effective if $n_Y \geq 0$ for all Y. If D is effective, we write $D \geq 0$.

Definition 118 (Complete Linear System of D)**.** *Given any $D \in \mathrm{Div}(X)$, define $|D| = \{D' \in \mathrm{Div}(X) : D' \sim D$ and $D' \geq 0\}$.*

Theorem 38. *If X is projective and $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then* $\dim_k \Gamma(X, \mathscr{F}) < \infty$.

Definition 119. *Let* $\ell(D) = \dim_k \Gamma(X, \mathscr{O}_X(D))$.

Theorem 39. $\mathbb{P}(\Gamma(X, \mathscr{O}_X(D))) \to |D|$ *by* $s \mapsto (s)$ *is bijective.*

Proof. Let $s \in \Gamma(X, \mathscr{O}_X(D))$, then $(s) \geq 0$ and $(s) \sim D$. So the map is well defined.

Injective: Suppose $s_1, s_2 \in \Gamma(X, \mathscr{O}_X(D))$, assume that $(s_1) = (s_2) \in \mathrm{Div}(X)$. Then $(s_1/s_2) = (s_1) - (s_2) = 0$, so $s_1/s_2 \in k[X] = k$.

Surjective: Let $D' \in |D|$. Then $D' \sim D$, so $D' = D + (f)$ where $f \in k(X)^*$. We define s to be the section given by $f \in \Gamma(X, \mathscr{O}_X(D))$. This is a global section, because $v_Y(f) \geq -n_Y$ for all Y, $(D = \sum n_Y[Y])$. Set $s_0 = 1 \in \Gamma(U, \mathscr{O}_X(D))$. $(s) = (f \cdot s_0) = (f) + (s_0) = (f) + D = D'$. ☐

Lemma 25. *X is a complete nonsingular curve, $D \in \mathrm{Div}(X)$, if $\ell(D) \neq 0$ then $\deg(D) \geq 0$ and if $\ell(D) \neq 0$ and $\deg(D) = 0$ then $D \sim 0$.*

Proof. $\ell(D) \neq 0$, then $|D| \neq \emptyset$, so $D \sim D' \geq 0$. So $\deg(D) = \deg(D') \geq 0$.

If $\deg(D) = 0$, then $\deg(D') = 0$, but as D' is effective, $D' = 0$. ☐

Chapter 12

Riemann-Roch Theorem

Let X be a complete nonsingular curve.

Definition 120 (Canonical Divisor). *$K \in \mathrm{Div}(X)$ is a canonical divisor on X if $\Omega_X \simeq \mathscr{O}(K)$.*

Definition 121 (Genus). *The genus $g = \ell(K) = \dim_k \Gamma(X, \Omega_X)$*

Example: $X = \mathbb{P}^1$, $\Omega_{\mathbb{P}^1} = \mathscr{O}(-2)$, so $g = 0$.
Example: $E = E_\lambda \subset \mathbb{P}^2$ elliptic curve, $\Omega_E \simeq \mathscr{O}_E$. So $g = 1$.

Theorem 40 (Riemann-Roch). *For any $D \in \mathrm{Div}(X)$ where X is a complete nonsingular curve, we have $\ell(D) + \ell(K - D) = \deg(D) + 1 - g$.*

Example: $X = \mathbb{P}^1$, $K = -2P$ for some $P \in \mathbb{P}^1$. The RRT theorem says that $\ell(nP)+\ell(-2P-nP) = n+1-0$, so if $n \geq 0$, we have that $\ell(nP) = n+1$. If $n = -1$, then $0 + 0 = -1 + 1 = 0$. If $n \geq -2$, we also see that it works.

Example: Set $D = K$, then $\ell(K) - \ell(K - K) = \deg(K) + 1 - g$, so $g - 1 = \deg(K) + 1 - g$, so $\deg(K) = 2g - 2$.

Corollary 25. *A nonsingular curve is either affine or projective.*

Proof. C nonsingular curve implies that $C = C_K \setminus \{P_1, \ldots, P_n\}$ where $K = k(C)$, $X = C_K$. If $m >> 0$, then $\ell(mP_i) = m + 1 - g \geq 2$. $1, f_i \in \Gamma(X, \mathscr{O}_X(mP_i))$, $f_i \notin k$.

$(f_i) = -r_i[P_i]$ effective divisor in $\mathrm{Div}(X)$. Set $f = \sum_{i=1}^n f_i \in k(X)^*$. f is defined exactly on $C \subseteq X$, so $C \simeq \mathrm{Spec}-m(\overline{k[f]})$ is affine. □

Corollary 26. *X is rational iff $g = 0$.*

Proof. $\Rightarrow$: genus of $\mathbb{P}^1$ is 0.

$\Leftarrow$: Let $P \neq Q \in X$. Set $D = P - Q \in \mathrm{Div}(X)$. By Riemann-Roch, $\ell(D) \geq \deg(D) + 1 - g$, so $\ell(D) \geq 1$. Thus $|D| \neq \emptyset$, so there is $D' \geq 0$ such that $D \sim D'$, but $\deg(D') = 0$, so $D' = 0$. $\square$

Corollary 27. *X complete nonsingular curve of $g = 1$, $P_0 \in X$, $\mathrm{char}(k) \neq 2$. Then $\exists$ isomorphism $X \simeq E_\lambda = V_+(zy^2 - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$ for some λ not 0 or 1, that sends $P_0 \mapsto (0:1:0)$.*

Proof. $\deg(K) = 2g-2 = 0$, so Riemann-Roch implies $\ell(nP_0) = n+1-1 = n$ for all $n \geq 1$, $k = \Gamma(\mathscr{O}_X) = \Gamma(\mathscr{O}_X(P_0)) \subsetneq \Gamma(\mathscr{O}_X(2P_0)) \subsetneq \ldots \subsetneq k(X)$.

Take $x \in \Gamma(\mathscr{O}_X(2P_0)) \setminus k$. $v_{P_0}(x) = -2$ $(x) = A + B - 2P_0$ for $A, B \in X$.

$x : X \to \mathbb{P}^1$ is a morphism, $x^*([0]-[\infty]) = A+B-2P_0$, so $x^*([0]) = A+B$, thus $[k(X) : k(x)] = \deg(x) = 2$.

Take $y \in \Gamma(\mathscr{O}_X(3P_0)) \setminus \Gamma(\mathscr{O}_X(2P_0))$. $v_{P_0}(y) = -3$, but as this is odd, $y \notin k(x)$. Thus $k(X) = k(x,y)$.

$\{1, x, y, x^2, xy\}$ is a basis for $\Gamma(\mathscr{O}_X(5P_0))$, $1, x, y, x^2, xy, x^3, y^2 \in \Gamma(\mathscr{O}_X(6P_0))$, $\dim = 6$.

So there exists a linear relations. Rescale x, y: $y^2 + b_1xy + b_0y = x^3 + a_2x^2 + a_1x + a_0$. Replace y with $y + \frac{1}{2}(b_1x + b_0)$, $y^2 = (x-a)(x-b)(x-c)$ where $a, b, c \in k$.

Claim: $a \neq b \neq c \neq a$.

$\varphi : X \setminus \{P_0\} \to C = V(y^2 - (x-a)(x-b)(x-c)) \subset \mathbb{A}^2$, $P \mapsto (x(P), y(P))$. This is birational, as $k(X) = k(x,y)$.

Assume $a = b = c$. Then C is a curve with a cusp, and is rational, so X would be rational, contradiction.

Assume $a = b \neq c$, then C is the nodal curve, which is also rational.

Replace x with $\frac{x-a}{b-a}$, rescale y, $y^2 - x(x-1)(x-\lambda)$ where $\lambda = \frac{c-a}{b-a} \neq 0, 1$.

$X \setminus \{P_0\} \to C \subset \mathbb{A}^2$ extends to an isomorphism $X \to E_\lambda$. $\square$

Lemma 26. *X complete nonsingular curve of genus g. Let $P_0, Q_0, \ldots, Q_g \in X$, then there exists $P_1, \ldots, P_g \in X$ such that $\sum_{i=0}^g P_i \sim \sum_{i=0}^g Q_i$.*

Proof. WLOG $P_0 \neq Q_i$ for all i.

Set $D = \sum Q_i$. $\ell(D) \geq \deg(D) + 1 - g = 2$. Thus, $\exists h \in \Gamma(X, \mathscr{O}_X(D))$ such that $h \notin k$. Set $f = h - h(P_0) \in k(X)^*$. $(f) = -D + P_0 + P_1 + \ldots + P_g = 0 \in \mathrm{C}\ell(X)$. $\square$

Corollary 28. *The Map $X^g = X \times \ldots \times X$ with g factors to $\mathrm{C}\ell^0(X)$ by $(P_1, \ldots, P_g) \mapsto \sum_{i=1}^g (P_i - P_0)$ is surjective.*

Proof. Note: Let $Q \in X$, the lemma implies that there are $P_1, \ldots, P_g \in X$ such that $(g+1)P_0 \sim Q + P_1 + \ldots + P_g$, so $-(Q - P_0) = \sum_{i=1}^{g}(P_i - P_0) \in \mathrm{C}\ell^0(X)$.

Let $D = \sum n_Q(Q - P_0) \in \mathrm{C}\ell^0(X)$. The note implies that we can assume $n_Q \geq 0$, and the lemma implies that we may assume $\sum n_Q \leq g$. $\square$

12.1 Blow-Up of Varieties

Let Y be an affine variety and $X \subset Y$ a closed subvariety. Write $I = I(X) = (f_0, \ldots, f_n) \subset k[Y]$. Then define the map $\varphi : Y \setminus X \to \mathbb{P}^n$ by $\varphi(y) = (f_0(y) : \cdots : f_n(y))$. We define the Blow-up of Y along X to be the closure of the graph of φ in $Y \times \mathbb{P}^n$.

Definition 122. *Set $B\ell_X(Y) = \overline{\{(y, \varphi(y)) \mid y \in Y \setminus X\}} \subset Y \times \mathbb{P}^n$, and let $\pi_1 : B\ell_X(Y) \to Y$ and $\pi_2 : B\ell_X(Y) \to \mathbb{P}^n$ denote the projections.*

The point of this construction is that if Y is singular along X, then usually $B\ell_X(Y)$ is less singular. Notice that since $\{(y, \varphi(y)) \mid y \in Y \setminus X\}$ is closed in $(Y \setminus X) \times \mathbb{P}^n$, this set agrees with $\pi_1^{-1}(Y \setminus X)$, so $\pi_1 : \pi_1^{-1}(Y \setminus X) \to Y \setminus X$ is an isomorphism. If $Y \setminus X$ is dense in Y, then $\pi_1 : B\ell_X(Y) \to Y$ is surjective; this follows because $\mathbb{P}^n$ is a complete variety. Notice also that if $Y \subset Y'$ is a closed subvariety, then $B\ell_X(Y) \subset B\ell_X(Y')$ is also a closed subvariety, while if $U \subset Y$ is an open subvariety, then $B\ell_{X \cap U}(U) \subset B\ell_X(Y)$ is an open subvariety.

The subvariety $E = \pi_1^{-1}(X) \subset B\ell_X(Y)$ is called the *exceptional divisor*. It is always a Cartier divisor in $B\ell_X(Y)$, with ideal sheaf isomorphic to $\pi_2^*(\mathscr{O}(1))$. To see this, let $k[z_0, \ldots, z_n]$ denote the homogeneous coordinate ring of $\mathbb{P}^n$ and set $J = I(B\ell_X(Y)) \subset k[Y][z_0, \ldots, z_n]$. Then $f_i z_j - f_j z_i \in J$ for all i, j. Set $\mathscr{L} = \pi_2^*(\mathscr{O}(-1))$ and define the global section $s = f_0/z_0 = f_1/z_1 = \cdots = f_n/s_n \in \Gamma(B\ell_X(Y), \mathscr{L})$. Then the exceptional divisor is the zero section $E = Z(s) \subset B\ell_X(Y)$.

Example 1. *Let $Y \subset \mathbb{A}^{n+1}$ be an closed subvariety containing the origin, and set $X = \{0\} \subset Y$. Then $I(X) = (x_0, \ldots, x_n) \subset k[Y] = k[\mathbb{A}^{n+1}]/I(Y)$ and the map $\varphi : Y \setminus X \to Y \times \mathbb{P}^n$ sends a point y to (y, ℓ), where ℓ is the line in $\mathbb{A}^{n+1}$ through 0 and y. It follows that $E \subset \mathbb{P}^n$ can be identified with the set of tangent directions of Y at the origin.*

Example 2. *Let* $Y = V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$ *and* $X = \{(0,0)\}$. *Then* $I(X) = (x,y) \subset k[Y]$ *and* $\varphi : Y \smallsetminus \{0\} \to \mathbb{P}^1$ *maps a point* P *to the line through* 0 *and* P. *We have* $B\ell_X(Y) = \{(P, \varphi(P)) \mid P \in Y \smallsetminus X\} \cup \{(0,(1 : -1)), (0,(1:1))\}$.

The variety $B\ell_X(Y)$ is independent of the chosen generators f_i of $I(X)$. It is a fact that the variety $B\ell_X(Y)$ can be recovered from the graded homogeneous coordinate ring $k[Y][z_0, \ldots, z_n]/J$. Therefore it is enough to show that $k[Y][z_0, \ldots, z_n]/J$ is isomorphic to the direct sum $\bigoplus_{d \geq 0} I^d$ of powers of I. This sum can be identified with the subring $\bigoplus_{d \geq 0} I^d t^d$ of $k[Y][t]$ generated by $k[Y]$ as well as $tf_0, \ldots, tf_n$. Define $\psi : Y \times \mathbb{A}^1 \to Y \times \mathbb{A}^{n+1}$ by $\psi(y,t) = (t, tf_0(y), \ldots, tf_n(y))$. Then $\overline{\psi(Y \times \mathbb{A}^1)}$ is the cone over $B\ell_X(Y)$ in $Y \times \mathbb{A}^{n+1}$, so $J = I(\psi(Y \times \mathbb{A}^1)) \subset k[Y][z_0, \ldots, z_n]$. Now $\psi^* : k[Y][z_0, \ldots, z_n] \to k[Y][t]$ satisfies $\psi^*(z_i) = tf_i$, $\operatorname{Ker}(\psi) = J$ and $\operatorname{Im}(\psi) = \bigoplus_{d \geq 0} I^d$, which establishes the isomorphism.

Let Y be an arbitrary variety and $X \subset Y$ a closed subvariety. Choose an open affine cover $Y = \cup Y_i$. Then the varieties $B\ell_{X \cap Y_i}(Y_i)$ can be glued together to form the blow-up $B\ell_X(Y) = \bigcup_i B\ell_{X \cap Y_i}(Y_i)$. This variety can also be constructed as $\operatorname{Proj}(\bigoplus_{d \geq 0} \mathscr{I}_X^d)$, where $\mathscr{I}_X \subset \mathscr{O}_Y$ is the ideal sheaf of X, and $\bigoplus_{d \geq 0} \mathscr{I}_X^d$ is the corresponding sheaf of graded $\mathscr{O}_Y$-algebras over Y.