Alg. Geo. 9/11/2006

$\mathbb{A}^n = k^n$ affine space.

$\mathbb{A}[\mathbb{A}^n] = k[x_1, \dots, x_n]$.

$I \subseteq \mathbb{A}[\mathbb{A}^n] \rightsquigarrow V(I) = \{ a \in \mathbb{A}^n \mid f(a) = 0 \ \forall f \in I \}$ alg. set.

$W \subseteq \mathbb{A}^n \rightsquigarrow I(W) = \{ f \in \mathbb{A}[\mathbb{A}^n] \mid f(a) = 0 \ \forall a \in W \}$.

W alg. set $\Rightarrow W = V(I(W))$.

In general: $\overline{W} = V(I(W))$ Zariski closure of W.

Nullstellensatz: $I(V(I)) = \sqrt{I} \subseteq \mathbb{A}[\mathbb{A}^n]$.

$\boxed{\text{Homework}}$: Will give weekly assignments.
Some of problems should be written up and turned in.
I will make careful comments to at least one of them.

<u>Serious people</u>:
Do all exercises, collaborate, come and ask questions!

Def A ring R is Noetherian if every ideal $I \subseteq R$ is f.g.

Exercise R Noetherian $\Leftrightarrow$ every ascending chain of ideals in R stab.
(if $I_1 \subseteq I_2 \subseteq \cdots$ ideals in R, then $I_N = I_{N+1} = \cdots$ for some N.)

Hilbert's basis Thm R Noetherian $\Rightarrow$ $R[X]$ Noetherian.
Has beautiful simple classical proof. Non-constructive $\Leftrightarrow$ controversial!

Corollary k field $\Rightarrow$ $k[x_1, \dots, x_n]$ Noetherian.

Def A topological space X is Noetherian if every descending chain of closed subsets stabilize.

Example $\mathbb{R}^n$ Not Noetherian. $B(0,1) \supsetneq B(0,\frac{1}{2}) \supsetneq B(0,\frac{1}{3}) \supsetneq \cdots$

Cor $\mathbb{A}^n$ is Noetherian. (Zariski top).

Proof $W_1 \supsetneq W_2 \supseteq W_3 \supseteq \cdots$ desc. chain of closed sets.
$I(W_1) \subseteq I(W_2) \subseteq I(W_3) \subseteq \cdots$ asc. chain of ideals in $k[x_1, \dots, x_n]$.
Must have $I(W_N) = I(W_{N+1}) = \cdots$
$\Rightarrow W_N = W_{N+1} = \cdots$
$\square$

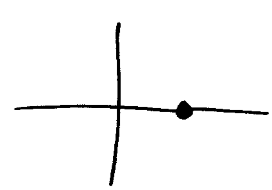
Thm Any closed subset of a Noeth. top space X is a union of finitely many irred. closed subsets.

Pf Suppose Thm false.
Then $\exists$ closed $W \subseteq X$ s.t. $W \neq$ union of fin. many irred. closed.
X Noetherian $\Rightarrow$ Can assume no proper subset of W has this property. $\boxed{\text{WHY?}}$

Now W Not irreducible, so $W = W_1 \cup W_2$, $W_i \subsetneq W$, W_i closed.
Assumption $\Rightarrow$ W_i = union of fin. many irred. closed.
$\Rightarrow$ W union of fin. many irred. closed. $\lightning$.
$\square$

Cor Every closed $W \subseteq \mathbb{A}^n$ is union of fin. many irred. closed subsets.

Example $\mathbb{A}^2$. $V(xy) = V(x) \cup V(y) \cup V(x-1, y)$.



Recall X top. space, $Y \subseteq X$ any subset.

Y has subspace topology: $U \subseteq Y$ open $\Leftrightarrow$ $\exists\, U' \subseteq X$ open s.t. $U = Y \cap U'$.

Note 1) $W \subseteq Y$ closed $\Leftrightarrow$ $W = \overline{W} \cap Y$

2) X Noetherian $\Rightarrow$ Y Noetherian.

$W_1 \supseteq W_2 \supseteq \cdots$ closed chain in Y

$\overline{W_1} \supseteq \overline{W_2} \supseteq \cdots$ closed chain in X, stabilizes!

$\overline{W_1} \cap Y \supseteq \overline{W_2} \cap Y \supseteq \cdots$ stabilizes in Y.

Def If $X \subseteq \mathbb{A}^n$ subset, the <u>Zariski topology on X</u> is the subspace top.

Def X any Noeth. top. space. The max. irred closed subsets of X are the (irreducible) <u>components</u> of X.

Exercises 1) X has finitely many irred. components.

2) X = union of its components.

3) $X \neq$ union of proper subset of its components.

Exercise A top. space is Noetherian
$\Updownarrow$
Every open subset is quasi-compact.

Exercise A Noetherian Hausdorf space is finite.

Recall: $X \subseteq \mathbb{A}^n$ closed set.

$A(X) = k[X_1, \ldots, x_n]/I(X)$ polynomial fcns on X.

Def If $f \in A(X)$, set $D(f) = \{a \in X \mid f(a) \neq 0\}$.

⊛ $D(f) \subseteq X$ is open.

Prop The sets $D(f)$ form a basis for the topology on X.

Proof Let $p \in U \subseteq X$, U open.

Must show $\exists\, f \in A(X)$: $p \in D(f) \subseteq U$.

$Z = X \smallsetminus U$ closed.

$Z \subsetneq Z \cup \{p\}$ $\Rightarrow$ $I(Z \cup \{p\}) \subsetneq I(Z)$.

Take any $f \in I(Z) \smallsetminus I(Z \cup \{p\})$.

$\square$

Regular functions

$X \subseteq \mathbb{A}^n$ alg. set, $\quad U \subseteq X$ open subset.

Def A fcn. $f : U \to k$ is called a <u>regular fcn</u> if f is locally rational.

I.e. $\exists$ open cover $U = \bigcup_\alpha U_\alpha$ and functions $p_\alpha, q_\alpha \in A(X)$ s.t. $q_\alpha(a) \neq 0$ and $f(a) = \frac{p_\alpha(a)}{q_\alpha(a)}$ $\forall a \in U_\alpha$.

Def $k[U] = \{\text{regular functions } U \to k\}$.

Note 1) $k[U]$ is a k-algebra. $\quad k = \{\text{constant fcns.}\}$

2) $A(X) \subseteq k[X]$.

Example $X = V(xy - zw) \subseteq \mathbb{A}^4$.

$U = D(y) \cup D(w) = X \setminus V(y, w)$ open subset.

Def. $f : U \to k$ by

$$f = \frac{x}{w} \text{ on } D(w) \quad \text{and} \quad f = \frac{z}{y} \text{ on } D(y).$$

This fcn. is regular, $f \in k[U]$.

Exercise $\nexists\, p, q \in A(X)$ s.t. $f(a) = \frac{p(a)}{q(a)}$ $\forall a \in U$.

(i.e. f is not globally rat. on U.)

Lemma Let $q_1, \ldots, q_m \in A(X)$.

Then $D(q_1) \cup \cdots \cup D(q_m) = X \iff (q_1, \ldots, q_m) = (1) = A(X)$.

Proof $\Leftarrow$: If $1 = \sum h_i q_i$ then at least one q_i must be $\neq 0$ at any point.

$\Rightarrow$: Take $Q_i \in k[x_1, \ldots, x_n]$ s.t. $q_i = \overline{Q_i} \in A(X)$.

$D(q_1) \cup \cdots \cup D(q_m) = X \Rightarrow$

$\emptyset = X \cap V(Q_1) \cap \cdots \cap V(Q_m) = V(I(X), Q_1, \ldots, Q_m)$

Weak NSS $\Rightarrow (I(X), Q_1, \ldots, Q_m) = (1) = k[x_1, \ldots, x_n]$

$\Rightarrow (q_1, \ldots, q_m) = (1) = A[X]$.

$\square$

Thm $X \subseteq \mathbb{A}^n$ any alg. subset. Then $k[X] = A(X)$.

Proof $f \in k[X]$ regular fcn.

$X = U_1 \cup \cdots \cup U_m$, $p_i, q_i \in A(X)$, $q_i \neq 0$ and $f = \frac{p_i}{q_i}$ on U_i

By refining open cover, can assume $U_i = D(g_i)$, $g_i \in A(X)$.

Note: $f = \frac{p_i}{q_i} = \frac{p_i g_i}{q_i g_i}$ on U_i and $D(g_i) = D(q_i g_i)$.

Replace p_i with $p_i g_i$, q_i with $q_i g_i$.

May assume $U_i = D(q_i) = D(q_i^2)$.

$X = D(q_1^2) \cup \cdots \cup D(q_m^2) \Rightarrow$

$$1 = \sum_{i=1}^{m} h_i q_i^2 \quad , \quad h_i \in A(X).$$

Now: $q_i^2 f = q_i p_i$. (This is true inside U_i; and $q_i = 0$ outside U_i.)

$$\therefore f = \sum_{i=1}^{m} h_i q_i^2 f = \sum_{i=1}^{m} h_i q_i p_i \in A(X).$$

$\square$

Weak Nullstellensatz $\Rightarrow$ $J = k[x_1, \dots, x_n, y]$. ⑤

$1 \in J \Rightarrow 1 = h_1 g_1 + \dots + h_m g_m + q \cdot (1 - yf)$ where $g_i \in I$, $h_i, q \in k[x_1, \dots, x_n, y]$.

Replace y with f^{-1} and multiply with f^N to clear denominators:

$$f^N = \tilde{h}_1 g_1 + \dots + \tilde{h}_m \cdot g_m + 0$$

where $\tilde{h}_i = f^N h_i(x_1, \dots, x_n, f^{-1}) \in k[x_1, \dots, x_n]$.

$\therefore$ $f^N \in I \Rightarrow f \in \sqrt{I}$.

$\square$

Note: $\{\text{radical ideals} \subseteq k[x_1, \dots, x_n]\} \longleftrightarrow \{\text{alg. subsets of } \mathbb{A}^n\}$

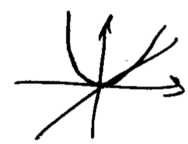
$I \longmapsto V(I)$

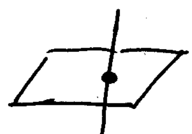
$I(W) \longleftarrow W$.

order reversing:

$I \subseteq J \Leftrightarrow V(I) \supseteq V(J)$ when I and J both radical.

Irreducible alg. sets.

Recall: $V(y^2 - xy - x^2 y + x^3) = V(y - x) \cup V(y - x^2) =$ 

$V(xz, yz) = V(x, y) \cup V(z)$ 

Def A (Zariski) closed set $W \subseteq \mathbb{A}^n$ is reducible if $W = W_1 \cup W_2$, W_i closed, $W_i \neq W$.

W is irreducible otherwise.

Prop $W \subseteq \mathbb{A}^n$ is irreducible $\Leftrightarrow$ $I(W)$ prime ideal.

Proof $\Rightarrow$: Suppose $I(W)$ not prime.

$\exists f_1, f_2 \notin I(W)$ s.t. $f_1 \cdot f_2 \in I(W)$.

Set $W_1 = W \cap V(f_1)$, $W_2 = W \cap V(f_2)$.

$f_i \notin I(W) \Rightarrow W \not\subseteq V(f_i) \Rightarrow W_i \subsetneq W$.

Claim: $W = W_1 \cup W_2$.

Let $a \in W$, $a \notin W_1$.

Then $f_1(a) \neq 0$.

$f_1 f_2 \in I(W) \Rightarrow 0 = (f_1 f_2)(a) = f_1(a) f_2(a) \Rightarrow f_2(a) = 0$.

$\therefore$ $a \in W_2$.

$\Leftarrow$: Assume $W = W_1 \cup W_2$ reducible, $W_i \subsetneq W$ closed.

Then $I(W_i) \supsetneq I(W)$.

Take $f_i \in I(W_i) \setminus I(W)$, $i = 1, 2$.

Then $f_1 f_2 \in I(W)$ so $I(W)$ not prime.

$\square$

Algebra $\longleftrightarrow$	Geometry
$k[x_1, \dots, x_n]$	$\mathbb{A}^n$
radical ideals	closed sets
prime ideals	irreducible closed sets
max. ideals	points.

Note: order reversing: $I \subseteq J \iff V(I) \supseteq V(J)$

when I and J both radical.

Alg. Geo. 9/18/2006

Def A SWF is a top space X with $U \mapsto k[U]$ = k-alg. of regular fcns.

(1) $U = \bigcup U_\alpha$: $f \in k[U] \Leftrightarrow f|_{U_\alpha} \in k[U_\alpha]$ $\forall \alpha$

(2) $f \in k[U] \Rightarrow D(f) \subseteq U$ open and $1/f \in k[D(f)]$.

Def A prevariety is a SWF X with open cover $X = U_1 \cup \cdots \cup U_m$, U_i affine.

Prop We have 1-1 correspondence

$$\{\text{affine alg. varieties}\} \longleftrightarrow \{\text{reduced f.g. } k\text{-algebras}\} \qquad \text{up to iso!!}$$
$$X \longleftrightarrow k[X]$$

Proof

Last time: Two affine vars. are isomorphic $\Leftrightarrow$ coord. rings isomorphic.

$\therefore$ The map $X \mapsto k[X]$ is injective.

Let R f.g. reduced k-alg., gen by $r_1, .., r_n$.

$k[x_1, .., x_n] \twoheadrightarrow R$, $x_i \mapsto r_i$

Set $I = \text{kernel}$, $X = V(I) \subseteq \mathbb{A}^n$.

$I = \sqrt{I} \Rightarrow A(X) = k[x_1, .., x_n]/I \cong R$.

$\square$

Note: Assume $m \subseteq R$ max. ideal. $\varphi : k[x_1, .., x_n] \twoheadrightarrow R$ as above.

Then $M := \varphi^{-1}(m) \subseteq k[x_1, .., x_n]$ max. ideal

$\Rightarrow M = (x_1 - a_1, .., x_n - a_n)$.

$\therefore R/m = k[x_1, .., x_n]/M = k$.

Canonical construction

R f.g. reduced k-algebra.

$\text{Spec-m}(R) = \{\text{max. ideals } m \subseteq R\}$.

Topology: Closed sets are $V(I) = \{m \supseteq I\}$

Let $f \in R$.

Def. $f : \text{Spec-m}(R) \longrightarrow k$, $f(m) = $ image of f by $R \longrightarrow R/m = k$.

(i.e. $f(m) \in k \subseteq R$ unique element s.t. $f - f(m) \in m$.)

If $U \subseteq \text{Spec-m}(R)$ open, $f : U \longrightarrow k$ function, then f is regular if it is locally of the form $f(m) = p(m)/q(m)$, $p, q \in R$.

Exercise $\text{Spec-m}(R) \cong X$, where $X \subseteq \mathbb{A}^n$ constructed in proof.

Subspaces of spaces with functions.

X SWF, $Y \subseteq X$ any subset.

Give Y structure of SWF as follows:

- Subspace topology
- If $U \subseteq Y$ open, $f : U \to k$ function, then f is regular if f can locally be extended to regular function on ~~X~~.

I.e. $\forall y \in U$ $\exists$ $U' \subseteq X$ open, and $F \in \mathcal{O}_X(U')$ s.t. $y \in U'$ and $f(x) = F(x)$ $\forall x \in U \cap U'$.

Exercises 1) Y is a SWF.

2) Inclusion $i : Y \to X$ is a morphism.

3) Let Z be a SWF and $\varphi : Z \to Y$ a function. Then φ morphism $\iff$ $i \circ \varphi : Z \to X$ morphism

4) The SWF structure on Y is determined by (2) and (3).

5) If $Z \subseteq Y \subseteq X$ then Z inherits same structure from X and Y.

Example $X \subseteq \mathbb{A}^n$ alg. set. X inherits its structure from $\mathbb{A}^n$.
If $Y \subseteq X$ closed, then Y's structure is inherited from X (or $\mathbb{A}^n$)

Prop A closed subset of a pre-variety is a pre-variety.

Proof X prevariety, $Y \subseteq X$ closed.

$X = U_1 \cup \cdots \cup U_m$, U_i open affine.

$Y \cap U_i$ closed in U_i $\Rightarrow$ $Y \cap U_i$ affine.

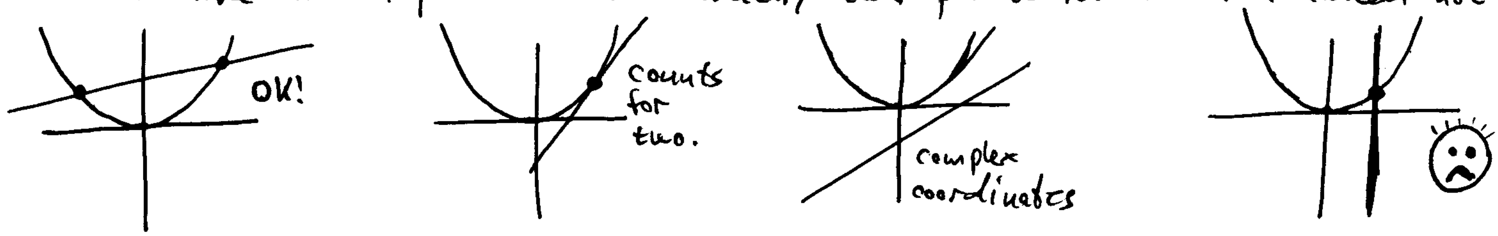
$Y = \bigcup_i Y \cap U_i$ $\Rightarrow$ Y pre-variety.

□

Projective Space

Thm Two distinct lines in the plane intersect in exactly one point. (if not parallel.)

Thm A line and a parabola have exactly two points in common. (when not false)



Solution: Add extra points at ∞ and make figures intersect there!
Intuition: two railroad rails meet at ∞.

<u>Def</u> Equiv. rel. $\sim$ on $\mathbb{A}^{n+1} \setminus \{0\}$:

$(a_0, a_1, \dots, a_n) \sim (\lambda a_1, \lambda a_2, \dots, \lambda a_n)$, $\lambda \in k^* = k \setminus \{0\}$.

Set $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\sim$, $\pi : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$ projection.

- Topology: $U \subseteq \mathbb{P}^n$ is open $\Leftrightarrow$ $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ is open.
- Reg. fcns: $f : U \to k$ is regular $\Leftrightarrow$ $\pi^*(f) = f \circ \pi : \pi^{-1}(U) \to k$ is regular.

$\mathbb{P}^n$ is a SWF called <u>projective space</u>.

<u>Note</u>: $\mathbb{P}^n = \{$ lines through origin in $\mathbb{A}^{n+1} \}$.

<u>Notation</u>: $(a_0 : a_1 : \dots : a_n) = \pi(a_0, a_1, \dots, a_n)$ = line through origin and $(a_0, \dots, a_n)$.

<u>Note</u>: If $f \in k[x_0, \dots, x_n]$ homogeneous of total deg. d, then

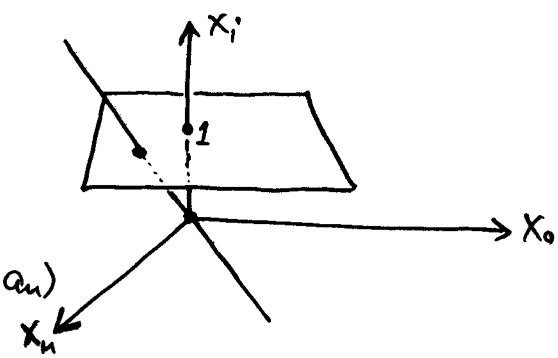
$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$.

$\therefore$ Well def. to ask if $f((a_0 : \dots : a_n)) = 0$, $\neq 0$.

<u>Def</u> $D_+(f) = \{ (a_0 : \dots : a_n) \in \mathbb{P}^n \mid f(a_0 : \dots : a_n) \neq 0 \}$.

<u>Thm</u> $\mathbb{P}^n$ is a pre-variety.

<u>Proof</u> Set $U_i = D_+(x_i)$, $0 \leq i \leq n$.



Claim: $U_i \cong \mathbb{A}^n$.

$\varphi : \mathbb{A}^n \longrightarrow U_i$

$(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \longmapsto (a_0 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n)$

$\psi : U_i \longrightarrow \mathbb{A}^n$, $(a_0 : \dots : a_n) \longmapsto \left(\frac{a_0}{a_i}, \dots, \widehat{\frac{a_i}{a_i}}, \dots, \frac{a_n}{a_i}\right)$.

$\mathbb{P}^n = U_0 \cup \dots \cup U_n$.

$\square$

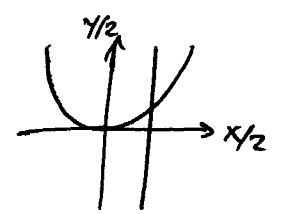
<u>Note</u>: $\mathbb{P}^n = D_+(x_0) \amalg V_+(x_0) = \mathbb{A}^n \amalg \mathbb{P}^{n-1}$.

$\mathbb{A}^n = \{ (1 : a_1 : \dots : a_n) \}$ usual n-space

$\mathbb{P}^{n-1} = \{ (0 : a_1 : \dots : a_n) \}$ points at ∞.

points at ∞ $\Leftrightarrow$ lines through origin in $\mathbb{A}^n$. $\leftrightarrow$ directions in $\mathbb{A}^n$.

<u>Example</u> Homogeneous coord ring of $\mathbb{P}^2 = k[\mathbb{A}^3] = k[x, y, z]$.



$\mathbb{A}^2 = D_+(z) \subseteq \mathbb{P}^2$. $k[\mathbb{A}^2] = k[\frac{x}{z}, \frac{y}{z}]$.

Closure of line: $\overline{V(\frac{x}{z} - 1)} = V_+(x - z) \subseteq \mathbb{P}^2$

Closure of parabola: $\overline{V(\frac{y}{z} - (\frac{x}{z})^2)} = V_+(yz - x^2)$.

$V_+(x - z) \cap V_+(yz - x^2) = \{ (1:1:1), (0:1:0) \}$

$(0:1:0)$ = point at ∞ in dir. of y-axi

<u>Exercise</u> · $k[\mathbb{P}^n] = k$.

<u>Exercise</u> X sWF, $\varphi : \mathbb{P}^n \to X$ function.

φ morphism $\Leftrightarrow$ $\varphi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X$ morphism.

Def $k[x_0, .., x_n]$ = projective coordinate ring of $\mathbb{P}^n$.

An ideal $I \subseteq k[x_0, .., x_n]$ is <u>homogeneous</u> if it is gen. by homogeneous polys.

Equivalent: $F = f_0 + f_1 + \cdots + f_d \in I$, f_i form of total deg i $\Rightarrow f_i \in I \ \forall i$.

<u>Def</u> For $W \subseteq \mathbb{P}^n$ subset, set $I(W) = I(\pi^{-1}(W)) \subseteq k[x_0, .., x_n]$.

<u>Note</u>: $I(W)$ is homogeneous: If $f = f_0 + \cdots + f_d \in I(W)$ and $(a_0 : \cdots : a_n) \in W$, then

$f(\lambda a_0, .., \lambda a_n) = f_0(a_0, .., a_n) + \cdots + \lambda^d f_d(a_0, .., a_n) = 0 \quad \forall \lambda \in k^*$

$\Rightarrow f_i(a_0, .., a_n) = 0 \quad \forall i$.

$\therefore f_i \in I(W)$.

<u>Def</u> If $I \subseteq k[x_0, .., x_n]$ <u>homogeneous</u> ideal, set

$V_+(I) = \{ (a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n \mid f(a_0, .., a_n) = 0 \quad \forall f \in I \}$.

<u>Thm</u> (Proj. Nullstellensatz).

$I \subseteq k[x_0, .., x_n]$ hom. ideal.

(a) $V_+(I) = \emptyset \Leftrightarrow (x_0, .., x_n)^N \subseteq I$ for some $N > 0$.

$\Leftrightarrow \sqrt{I} = (1)$ or $\sqrt{I} = (x_0, .., x_n)$.

(b) $V_+(I) \neq \emptyset \Rightarrow I(V_+(I)) = \sqrt{I}$.

<u>Proof</u> (a) If $V_+(I) = \emptyset$ then $V(I) = \emptyset$ or $V(I) = \{0\} \subseteq \mathbb{A}^{n+1}$.

$\Rightarrow \sqrt{I} = I(V(I)) = (1)$ or $\sqrt{I} = (x_0, .., x_n)$.

(b) $V_+(I) \neq \emptyset \Rightarrow \overline{\pi^{-1}(V_+(I))} = \pi^{-1}(V_+(I)) \cup \{0\} = V(I) \subseteq \mathbb{A}^{n+1}$.

$\Rightarrow I(V_+(I)) = I(V(I)) = \sqrt{I}$. □

<u>Note</u>: $\left\{ \begin{array}{c} \text{closed subsets} \\ \text{of } \mathbb{P}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{homogeneous radical} \\ \text{ideals in } k[x_0, .., x_n] \\ \underline{\text{except}} \ (x_0, .., x_n) \end{array} \right\}$

"irrelevant ideal"

Def . X top. space, $W \subseteq X$ subset.
W is locally closed if $W =$ open $\cap$ closed $\subseteq X$.

Note: A locally closed subset of a pre-variety is a prevariety.

Terminology

projective variety $=$ closed subset of $\mathbb{P}^n$
quasi-projective var. $=$ locally closed $\subseteq \mathbb{P}^n$
affine variety $=$ closed $\subseteq \mathbb{A}^n$
quasi-affine var. $=$ locally closed $\subseteq \mathbb{A}^n$.

Example (quasi) affine $\Rightarrow$ quasi-projective.

Exercise $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Later: X both projective and quasi-affine $\Rightarrow$ X finite.

Def Let $X \subseteq \mathbb{P}^n$ closed, proj. variety.
Proj. coord. ring of X $= k[x_0, \dots, x_n]/I(X)$.

Warning proj. coordinate ring depends on embedding of X in $\mathbb{P}^n$!

Example $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$, $\varphi(a:b) = (a^2 : ab : b^2)$.

$\varphi : \mathbb{P}^1 \xrightarrow{\cong} V_+(xz - y^2) \subseteq \mathbb{P}^2$.

$k[s,t] \not\cong k[x,y,z]/(xz - y^2)$.

HW Back: I like when people realize that they can't prove a statement, as opposed to turning in flawed argument.

Products

X and Y are sets. $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$.

~~A product of~~ What is $X \times Y$ really?

$X \times Y$ is a set with projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$.

Univ property: If Z any set with functions $f : Z \to X$, $g : Z \to Y$ then $\exists!\ \varphi : Z \to X \times Y$ s.t. $f = \pi_X \circ \varphi$, $g = \pi_Y \circ \varphi$.

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "f"] \arrow[r, dashed, "\exists!\varphi"'] \arrow[drr, bend right, "g"'] & X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"] & \\ & & Y \end{tikzcd}$$

Def Let X, Y be SWFs. A product of X and Y is a SWF "$X \times Y$" with morphisms $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ satisfying univ prop: If Z any SWF and $f : Z \to X$, $g : Z \to Y$ morphisms, then $\exists!$ morphism $\varphi : Z \to X \times Y$ s.t. $f = \pi_X \varphi$, $g = \pi_Y \varphi$.

Exercise Assume $(P, \pi_X : P \to X, \pi_Y : P \to Y)$ and $(P', \pi'_X : P' \to X, \pi'_Y : P' \to Y)$ are both products of X and Y. Then $\exists!$ iso $\varphi : P \xrightarrow{\cong} P'$ s.t. $\pi_X = \pi'_X \circ \varphi$, $\pi_Y = \pi'_Y \circ \varphi$.

Example $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. Note: NOT product topology!!!

Construction $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$ as a set. (False for schemes)

If $U \subseteq X$ open, $V \subseteq Y$ open then $U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V) \subseteq X \times Y$ must be open!

Let $g_1, \dots, g_n \in \mathcal{O}_X(U)$, $h_1, \dots, h_n \in \mathcal{O}_Y(V)$. Set $f(u,v) = \sum_{i=1}^{n} g_i(u)\, h_i(v)$.

Then $f : U \times V \to k$ must be regular!

So $D_{U \times V}(f) = \{(u,v) \in U \times V \mid f(u,v) \neq 0\}$ must be open in $X \times Y$!

Topology: $S \subseteq X \times Y$ open $\iff$ $S =$ union of sets $D_{U \times V}(f)$.

Regular fcns: $F : S \to k$ is regular if it can locally be written as $\dfrac{f'(u,v)}{f(u,v)}$ on some $D_{U \times V}(f)$.

I.e.: $\forall\ (x,y) \in S$ $\exists$ $U \subseteq X$ open, $V \subseteq Y$ open, and $f(u,v) = \sum g_i(u) h_i(v)$, $f'(u,v) = \sum g'_j(u) h'_j(v)$, with $g_i, g'_j \in \mathcal{O}_X(U)$, $h_i, h'_j \in \mathcal{O}_Y(V)$ such that $(x,y) \in D_{U \times V}(f) \subseteq S$ and $F(u,v) = \dfrac{f'(u,v)}{f(u,v)}$ $\forall\ (u,v) \in D_{U \times V}(f)$.

Exercises
1) $X \times Y$ is a SWF
2) $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are morphisms.
3) $X \times Y =$ product of X and Y.

Remark X, Y SWFs, $U \subseteq X$, $V \subseteq Y$ any subsets.
Then U and V have inherited structures of SWF.
$U \times V$ has product SWF-structure and inherited SWF-structure $U \times V \subseteq X \times Y$.
These are the same by universal properties!

$$\begin{tikzcd} & & U \arrow[r] & X \\ Z \arrow[r, "\varphi"] & U \times V \arrow[ur] \arrow[dr] \arrow[r, dashed, "i"', "\text{morph}"] & X \times Y \arrow[ur] \arrow[dr] & \\ & & V \arrow[r] & Y \end{tikzcd}$$

$U \times V =$ product.
φ morphism $\Leftrightarrow$ $i \circ \varphi$ morphism.

Def A SWF X is *separated* if $\forall$ SWF Y and morphisms $f, g : Y \to X$, the set $\{y \in Y \mid f(y) = g(y)\} \subseteq Y$ is closed.

Example $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{0_1, 0_2\}$

Def. $\varphi_i : \mathbb{A}^1 \to X$, $\varphi_i(a) = \begin{cases} a & \text{if } a \neq 0 \\ 0_i & \text{if } a = 0 \end{cases}$

Topology: $U \subseteq X$ open $\Leftrightarrow$ $\varphi_i^{-1}(U) \subseteq \mathbb{A}^1$ open $\forall i$

Reg. Fcns: $f : U \to k$ regular $\Leftrightarrow$ $\varphi_i^*(f) = f \circ \varphi_i : \varphi_i^{-1}(U) \to k$ regular $\forall i$.

$X =$ prevariety: $X = \varphi_1(\mathbb{A}^1) \cup \varphi_2(\mathbb{A}^1)$ open cover, and $\varphi_i(\mathbb{A}^1) \cong \mathbb{A}^1$.

X is NOT separated: $\{a \in \mathbb{A}^1 : \varphi_1(a) = \varphi_2(a)\} = \mathbb{A}^1 \setminus \{0\}$ not closed.

Def An *algebraic variety* is a separated prevariety.

Exercise (a) Any subspace of a separated SWF is separated.
(b) Any product of separated SWFs is separated.

Remark X any SWF. Then $\Delta : X \to X \times X$, $x \mapsto (x,x)$ is a morphism.
Set $\Delta_X = \Delta(X) \subseteq X \times X$. Then $\Delta : X \xrightarrow{\cong} \Delta_X \subseteq X \times X$.

Lemma X separated $\Leftrightarrow$ $\Delta_X \subseteq X \times X$ closed

Proof $\Rightarrow$: $\pi_i : X \times X \to X$ projections, $i = 1, 2$.
$\Delta_X = \{z \in X \times X \mid \pi_1(z) = \pi_2(z)\} \subseteq X \times X$ is closed.

$\Leftarrow$: Let $f, g : Y \to X$ be morphisms.
Def. $\varphi : Y \to X \times X$ by $\varphi(y) = (f(y), g(y))$. Then φ morphism.
Δ_X closed $\Rightarrow$ $\{y \in Y \mid f(y) = g(y)\} = \varphi^{-1}(\Delta_X)$ is closed. □

Exercise A top. space is Hausdorff $\Leftrightarrow$ $\Delta_X \subseteq X \times X$ is closed.
NB: with product topology!

Exercise $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$

Prop All affine varieties are separated.

Proof Enough to show $\mathbb{A}^n$ separated, i.e. $\Delta_{\mathbb{A}^n} \subseteq \underbrace{\mathbb{A}^n \times \mathbb{A}^n}_{= \mathbb{A}^{2n}}$ closed.

~~[illegible]~~ Write $k[\mathbb{A}^{2n}] = k[x_1, \dots, x_n, y_1, \dots, y_n]$.

$\Delta_{\mathbb{A}^n} = V(x_1 - y_1, \dots, x_n - y_n) \subseteq \mathbb{A}^{2n}$ closed.

$\square$

Products of affine varieties.

Lemma Let A, B f.g. reduced k-algebras, $k=\bar{k}$.
Then $A\otimes_k B$ f.g. reduced k-algebra.

Recall: ring structure on $A\otimes B$ given by $(a_1\otimes b_1)\cdot(a_2\otimes b_2) = a_1a_2\otimes b_1b_2$.

Proof A gen by $a_1,\dots,a_n$, B gen by $b_1,\dots,b_m$.
Then $A\otimes B$ gen. by $a_1\otimes 1,\dots,a_n\otimes 1$, $1\otimes b_1,\dots,1\otimes b_m$. So f.g.
Let X and Y be affine vars. with $k[X]=A$, $k[Y]=B$.

Def. $\phi : A\otimes_k B \longrightarrow \{\text{functions } X\times Y\to k\}$ ring hom.
$f\otimes g \longmapsto [(x,y)\mapsto f(x)g(y)]$.

Claim: ϕ is injective. ($\Rightarrow A\otimes B$ is reduced!)

Suppose $\phi(\sum_{i=1}^n f_i\otimes g_i) = 0$.

WLOG: $g_1,\dots,g_n\in B$ linearly indep. over k.

If $x\in X$ then $\sum_{i=1}^n f_i(x)\, g_i = 0\in B=k[Y]$.

$\{g_i\}$ lin. indep $\Rightarrow f_i(x)=0 \ \forall i$

$\therefore f_i = 0\in A=k[X]$.

□

Thm (a) If X and Y are affine vars. then so is $X\times Y$, and $k[X\times Y]=k[X]\otimes_k k[Y]$.
(b) If X, Y prevarieties then $X\times Y$ is a prevariety.

Proof Set $P=\text{Spec-m}(k[X]\otimes_k k[Y])$

Have k-alg. homs. $k[X]\longrightarrow k[X]\otimes k[Y]$, $f\longmapsto f\otimes 1$ and $k[Y]\longrightarrow k[X]\otimes k[Y]$, $g\mapsto 1\otimes g$

Give morphisms $\pi_X : X\times Y\to X$ and $\pi_Y : X\times Y\longrightarrow Y$.

Claim: $P = X\times Y$.

Let Z any SuF and $p: Z\to X$, $q: Z\to Y$ morphisms.

Def k-alg. hom. $k[X]\otimes k[Y]\longrightarrow k[Z]$, $f\otimes g\mapsto p^*(f)\cdot q^*(g)$.

This defines morphism $\varphi: Z\longrightarrow P$, and φ is unique.

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "p"] \arrow[r, dashed, "\exists!\varphi"'] \arrow[drr, bend right, "q"'] & P \arrow[ur] \arrow[dr] & \\ & & Y \end{tikzcd}$$

(b) $X=\bigcup U_i$, $Y=\bigcup V_j$ open affine coverings. $X\times Y = \bigcup_{i,j} U_i\times V_j$ open cover, each $U_i\times V_j$ is affine.

□

Remark If Y affine, $X \subseteq Y$ closed, then $k[X] = k[Y]/I(X)$, so $i^* : k[Y] \to k[X]$ surjective. $(i : X \to Y)$

On the other hand: Assume X, Y affine, $\varphi : X \to Y$ morphism, $\varphi^* : k[Y] \twoheadrightarrow k[X]$ surj.

Set $I = \ker(\varphi^*)$.

$$\begin{tikzcd} k[Y] \arrow[r, "\varphi^*"] \arrow[dr] & k[X] \\ & k[Y]/I \arrow[u, "\cong"'] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} Y & X \arrow[l, "\varphi"'] \arrow[d, "\cong"] \\ & V(I) \arrow[ul, phantom, "\supseteq"] \end{tikzcd}$$

$\therefore$ $X \subseteq Y$ closed subvar.

Prop A prevariety X is separated iff $\forall$ open affines $U, V \subseteq X$: $U \cap V$ affine and $k[U] \otimes_k k[V] \twoheadrightarrow k[U \cap V]$ surj.

Proof $\boxed{\text{[map given by } k[U]\otimes k[V] = k[U\times V] \to k[\Delta_{U\cap V}] \Rightarrow k[U\cap V]\text{]}}$

$\Rightarrow$: $U \cap V \cong \Delta_{U\cap V} = \Delta_X \cap (U \times V) \subseteq U \times V$ closed.

$U \times V$ affine $\Rightarrow$ $U \cap V$ affine, and $k[U \times V] \twoheadrightarrow k[U \cap V]$ surj.

$\Leftarrow$: If $U, V, U\cap V$ affine and $k[U \times V] \to k[U \cap V]$ surjective, then $U \cap V \xrightarrow{\Delta} U \times V$ inclusion of closed subset

$\Rightarrow$ $\Delta_X \cap (U \times V) \subseteq U \times V$ closed.

$X \times X = \bigcup (U \times V)$ open cover $\Rightarrow$ $\Delta_X \subseteq X \times X$ closed.

$\square$

Exercise (a) Let X be a prevariety satisfying:
$\forall x, y \in X$ $\exists$ open affine $U \subseteq X$ s.t. $x, y \in U$.
Then X is separated.
(b) $\mathbb{P}^n$ has this property.

Corollary All quasi-projective varieties are separated.

Products of Projective varieties are projective

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be closed.
Then $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed.
Enough: $\mathbb{P}^n \times \mathbb{P}^m$ is projective, i.e. $\mathbb{P}^n \times \mathbb{P}^m \subseteq \mathbb{P}^N$ closed.

Segre map

$N = (n+1)(m+1) - 1 = mn + n + m$.

$s : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$, $(x_0 : \cdots : x_n) \times (y_0 : \cdots : y_m) \mapsto (x_0y_0 : x_0y_1 : \cdots : x_0y_m : x_1y_0 : \cdots\cdots : x_ny_m)$.

Call the proj. coords on $\mathbb{P}^N$ for z_{ij}, $0 \le i \le n$, $0 \le j \le m$.

Exercise $s : \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\cong} V_+(\{z_{ij}z_{kl} - z_{il}z_{kj}\}) \subseteq \mathbb{P}^N$.

Alternative proof that $\mathbb{P}^n$ separated:

$\Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n \subseteq \mathbb{P}^N$, $\quad N = n^2 + 2n$.

Exercise: $\Delta_{\mathbb{P}^n} = V_+(\{z_{ij} - z_{ji}\}) \subseteq \mathbb{P}^N$ is closed.

Last time: X and Y affine varieties $\Rightarrow$ $X \times Y$ affine, and $k[X\times Y] = k[X] \otimes_k k[Y]$,
$\Rightarrow$ products of (pre) varieties are (pre) varieties.

Remark If Y affine, $X \subseteq Y$ closed, then $k[X] = k[Y]/I(X)$, so
$i^* : k[Y] \longrightarrow k[X]$ surjective, $(i : X \to Y)$.
On the other hand: Assume X, Y affine, $\varphi : X \to Y$ morphism, $\varphi^* : k[Y] \twoheadrightarrow k[X]$ surj.
Set $I = \ker(\varphi^*)$.

$$\begin{tikzcd} k[Y] \arrow[r, "\varphi^*"] \arrow[dr] & k[X] \\ & k[Y]/I \arrow[u, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} Y & & X \arrow[ll, "\varphi"'] \arrow[dl, "\cong"] \\ & V(I) \arrow[ul] & \end{tikzcd}$$

$\therefore$ $X \subseteq Y$ closed subvariety.

Recall: $\Delta : X \to X \times X$, $x \mapsto (x,x)$.
$\Delta_X = \{(x,x) \in X \times X\}$.

Prop A prevariety X is separated if and only if
$\forall$ open affines $U, V \subseteq X$: $U \cap V$ is affine, and $k[U] \otimes k[V] \twoheadrightarrow k[U\cap V]$ surj.
Map given by $k[U] \otimes k[V] = k[U \times V] \xrightarrow[\text{restrict}]{} k[\Delta_{U\cap V}] = k[U \cap V]$.

Proof
$\Rightarrow$: $U \cap V \cong \Delta_{U \cap V} = \Delta_X \cap (U \times V) \subseteq U \times V$ closed.
$\therefore$ $U \cap V$ affine and $k[U \times V] \twoheadrightarrow k[U \cap V]$ surj.
$\Leftarrow$: If U, V, $U \cap V$ affine and $k[U\times V] \to k[U \cap V]$ surjective
then $\Delta : U \cap V \longrightarrow U \times V$ inclusion of closed subset
$\Rightarrow$ $\Delta_X \cap (U \times V) \subseteq U \times V$ closed.
$X \times X = \bigcup (U \times V)$ open cover $\Rightarrow$ $\Delta_X \subseteq X \times X$ closed.
$\square$

Exercises (a) Let X be a prevar. satisfying:
$\forall\, x, y \in X$ $\exists$ open affine $U \subseteq X$ s.t. $x, y \in U$.
Then X is separated.
(b) $\mathbb{P}^n$ has this property.

Cor All proj. varieties are separated.

Products of projective vars are projective.
Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be closed.
Then $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed.
Enough: $\mathbb{P}^n \times \mathbb{P}^m$ is projective, i.e. $\mathbb{P}^n \times \mathbb{P}^m \subseteq \mathbb{P}^N$ closed.

Segre map $N = (n+1)(m+1) - 1 = nm + n + m$
$S : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$, $(x_0 : \dots : x_n) \times (y_0 : \dots : y_m) \mapsto (x_0y_0 : x_0y_1 : \dots : x_0y_m : x_1y_0 : \dots : x_ny_m)$
Call proj. coords on $\mathbb{P}^N$ for z_{ij}, $0 \le i \le n$, $0 \le j \le m$.

Exercise $S : \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\cong} V_+(\{z_{ij} z_{pq} - z_{iq} z_{pj}\}) \subseteq \mathbb{P}^N$.

Note: $\mathbb{P}^n \times \mathbb{P}^n \subseteq \mathbb{P}^N$ closed, $N = n^2 + 2n$.

Exercise $\Delta_{\mathbb{P}^n} = V_+(\{z_{ij} - z_{ji}\}) \subseteq \mathbb{P}^N$ is closed. [redacted] – other way to show proj. varieties are separated.

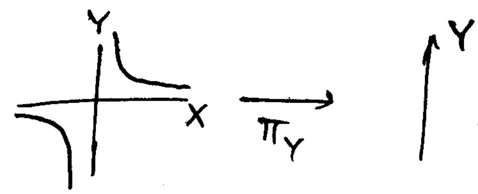
Complete Varieties ($\leftrightarrow$ compact manifolds.)

Def A variety X is complete if for any variety Y, the projection
$\pi_Y : X \times Y \longrightarrow Y$ is closed.
I.e. $Z \subseteq X \times Y$ closed $\Rightarrow$ $\pi_Y(Z) \subseteq Y$ closed.

Note: 1) Closed subsets of complete varieties are complete.
2) Products of complete varieties are complete.

Example Points are complete!

Example $X = \mathbb{A}^1$ is NOT complete.
$Y = \mathbb{A}^1$, $Z = V(xy - 1) \subseteq \mathbb{A}^2 = X \times Y$.
$\pi_Y(Z) = \mathbb{A}^1 \setminus \{0\}$ not closed.



Prop $\varphi : X \longrightarrow Y$ morphism of varieties.
If X complete, then $\varphi(X)$ is complete, and closed in Y.

Proof
$\Gamma(\varphi) := \{(x, \varphi(x)) : x \in X\} \subseteq X \times Y$.
$= (\varphi \times 1)^{-1}(\Delta_Y)$ where $\varphi \times 1 : X \times Y \longrightarrow Y \times Y$.
Y separated $\Rightarrow$ $\Gamma(\varphi) \subseteq X \times Y$ closed.
X complete $\Rightarrow$ $\varphi(X) = \pi_Y(\Gamma(\varphi)) \subseteq Y$ closed.
Let $Z \subseteq \varphi(X) \times Y'$ be closed.
Then $W = (\varphi \times 1)^{-1}(Z) \subseteq X \times Y'$ closed.

$$\begin{tikzcd} X \times Y' \arrow[r, "\varphi\times 1"] \arrow[dr, "\tilde{\pi}_{Y'}"'] & \varphi(X) \times Y' \arrow[d, "\pi_{Y'}"] \\ & Y' \end{tikzcd}$$

$\Rightarrow \pi_{Y'}(Z) = \pi_{Y'}((\varphi \times 1)(W)) = \tilde{\pi}_{Y'}(W)$ closed in Y'.
$\square$

~~Prop X complete irred. variety ⇒ k[X] = k~~

Exercise $\varphi : X \to Y$ ~~morphism of varieties~~ cont. map of top spaces. X irreducible $\Rightarrow$ $\overline{\varphi(X)}$ irreducible top space.

Prop X complete irred variety $\Rightarrow$ $k[X] = k$.

Proof Let $f \in k[X]$. Then $f : X \longrightarrow \mathbb{A}^1 = k$ morphism.
$f(X) \subseteq \mathbb{A}^1$ complete irred. closed subset.
$\Rightarrow f(X) = $ single point $\Rightarrow$ f constant.
$\square$

Cor A complete quasi-affine variety has finitely many points.

Proof Let X be such a variety.
WLOG X irreducible.
If $X \subseteq \mathbb{A}^n$ locally closed, then $\pi_i : X \to k$ constant $\Rightarrow X = \{\text{point}\}$.
$\square$

Thm ~~[illegible]~~ $\mathbb{P}^n$ is complete.
In particular, X projective $\Rightarrow k[X] = k$ and X not quasi-affine.

Note: $I \subseteq S = k[x_0, \dots, x_n]$ hom. ideal.
Then $V_+(I) \neq \emptyset \iff I_d \subsetneq S_d \quad \forall d \in \mathbb{N}$.

$\Rightarrow$: If $x \in V_+(I)$ then $I_d \subseteq I(V_+(I))_d \subseteq I(\{x\})_d \subsetneq S_d$.

$\Leftarrow$: If $V_+(I) = \emptyset$ then $\sqrt{I} \supseteq (x_0, \dots, x_n)$ by proj. Nullstellensatz
$\Rightarrow x_i^m \in I$ for some m, all i, $\Rightarrow I_d = S_d \quad \forall d \geq (n+1)m$.

Proof Y variety, $Z \subseteq \mathbb{P}^n \times Y$ closed. Show $\pi_Y(Z) \subseteq Y$ closed.
$Y = \bigcup Y_i$ open affine cover.
Enough to show $\pi_Y(Z) \cap Y_i$ closed in $Y_i \quad \forall i$
$\pi_Y(Z) \cap Y_i = \pi_{Y_i}(Z \cap (\mathbb{P}^n \times Y_i))$, $\quad \pi_{Y_i} : \mathbb{P}^n \times Y_i \to Y_i$.

WLOG: Y affine.

$C(Z) := \overline{(\pi \times id)^{-1}(Z)} \subseteq \mathbb{A}^{n+1} \times Y$, $\quad \pi \times id : (\mathbb{A}^{n+1} \setminus \{0\}) \times Y_i \to \mathbb{P}^n \times Y_i$

$k[\mathbb{A}^{n+1} \times Y] = S \otimes_k k[Y] = k[Y][x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d \otimes_k k[Y]$ graded ring. $\deg(x_i) = 1$.

Note: $(y, (a_0, \dots, a_n)) \in C(Z) \Rightarrow (y, (\lambda a_0, \dots, \lambda a_n)) \in C(Z) \quad \forall \lambda \in k$.
$\Rightarrow I(C(Z)) \subseteq k[\mathbb{A}^{n+1} \times Y]$ homogeneous ideal.

Write $I(C(Z)) = (f_1, \dots, f_m)$ where $f_i \in S_{d_i} \otimes k[Y]$.

For $y \in Y$, $f_i(y) := f_i(-, y) \in S_{d_i}$

OBS: $y \in \pi_Y(Z) \iff \exists x \in \mathbb{P}^n$ s.t. $(x, y) \in Z$
$\iff V_+(f_1(y), \dots, f_m(y)) \neq \emptyset \subseteq \mathbb{P}^n$
$\iff (f_1(y), \dots, f_m(y))_d \neq S_d \quad \forall d \geq 0$.

Fix $d \geq 0$.
$y \in Y$ def. linear map $\bigoplus_{i=1}^m S_{d-d_i} \longrightarrow S_d \quad (g_1, \dots, g_m) \mapsto \sum_{i=1}^m g_i \cdot f_i(y)$.
Each entry in matrix of this map is a regular function in y. !

Now: $(f_1(y), \dots, f_m(y))_d \neq S_d \iff \operatorname{rank}(\text{matrix}) < \dim_k(S_d) = \binom{n+d}{d}$ 4

$\iff$ all minors of size $\binom{n+d}{d}$ vanish.

$\therefore\ W_d := \{ y \in Y \mid (f_1(y), \dots, f_m(y))_d \neq S_d \} \subseteq Y$ closed.

$\pi_Y(Z) = \bigcap_{d \geq 0} W_d \subseteq Y$ also closed.

$\square$

Challenge: Find a complete variety that is not projective.

~~Rational maps.~~
~~Idea: A morphism $f: X \to Y$ between~~

Exercise X top space, $W \subseteq X$ any subset.

(a) $\overline{W} = X \iff W \cap U \neq \emptyset \quad \forall\ \emptyset \neq U \subseteq X$ open.

(b) $f: X \to Y$ cont., $\overline{f(X)} = Y$, $\overline{W} = X \Rightarrow \overline{f(W)} = Y$.

(c) X irred. and $\emptyset \neq U \subseteq X$ open $\Rightarrow \overline{U} = X$ and U irred.

Rational maps

~~Idea: A morphism $f: X \to Y$ between irred. varieties is uniquely determined by its restriction to any nonempty open $\subseteq X$.~~

X, Y irred. varieties.

Idea: A morphism $f: X \to Y$ is uniquely determined by $f|_U$ if $\emptyset \neq U \subseteq X$ open.

Consider pairs (U, f) s.t. $\emptyset \neq U \subseteq X$ open, $f: U \to Y$ morphism.

Relation: $(U, f) \sim (V, g) \iff f|_{U \cap V} = g|_{U \cap V}$.

Y separated and X irred $\Rightarrow \sim$ equiv. relation.

$(U, f) \sim (V, g) \sim (W, h) \Rightarrow f = g = h$ on $U \cap V \cap W \Rightarrow f = h$ on $U \cap W$.

Def A rational map $f: X \dashrightarrow Y$ or $(U, f): X \dashrightarrow Y$ is an equivalence class for $\sim$.

Remark $f: X \dashrightarrow Y$ rat. map. Then $\exists$ max. open $U \subseteq X$ where f is def. as a function.

$U =$ union of all $V \subseteq X$ for which $f \sim (V, g)$.

Example $f: \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$, $f(x,y) = (x/y, y/x^2)$ is def. on $D(xy) \subseteq \mathbb{A}^2$.

Exercise $f: \mathbb{A}^2 \dashrightarrow \mathbb{A}^2 \subseteq \mathbb{P}^2$.

Find max open set where $f: \mathbb{A}^2 \dashrightarrow \mathbb{P}^2$ is defined.

Def A <u>rational function</u> on X is a rational map $f: X \dashrightarrow \mathbb{A}^1$.
f is given by a regular function $f: U \to k$, $\emptyset \neq U \subseteq X$ open.
$k(X) = \{ f : X \dashrightarrow k \}$ field of rat. fcns. on X.

Note: If (U, f), $(V, g) \in k(X)$, then $f+g$, $f-g$, $fg : U \cap V \to k$ rat. fcns.
If $f \neq 0$ then $1/f : D(f) \to k$ rat. fcn. on X.
$\therefore$ $k(X)$ is a field.

Example $k(\mathbb{A}^n) = k(x_1, \dots, x_n)$.

$k(\mathbb{P}^n) = k\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = \left\{ p/q \mid p, q \in k[x_0, \dots, x_n] \text{ forms of same degree} \right\}$.

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X, Y irred. varieties.

Def A <u>rational map</u> $f: X \dashrightarrow Y$ is an equiv. class of pairs (U, f) where $\emptyset \neq U \subseteq X$ open, $f: U \to Y$ morphism.

$(U, f) \sim (V, g) \Leftrightarrow f|_{U\cap V} = g|_{U \cap V}$.

Note: $(U, f) \sim (V, g) \sim (W, h) \Rightarrow f = h$ on $U \cap V \cap W$.

X irred $\Rightarrow U \cap V \cap W$ dense open in $U \cap W$.

Y separated $\Rightarrow \{x \in U \cap W : f(x) = h(x)\}$ is closed.

$\therefore f = h$ on $U \cap W$.

Remark If $f: X \dashrightarrow Y$ then $\exists$ max. open $U \subseteq X$ where f def. as morph.

$U = \bigcup_{(V,g)\sim f} V \subseteq X$.

Example $f: \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$, $f(x,y) = (x/y, y/x^2)$ is def. on $D(xy) \subseteq \mathbb{A}^2$.

Exercise $f: \mathbb{A}^2 \dashrightarrow \mathbb{A}^2 \subseteq \mathbb{P}^2$. Find open set where $f: \mathbb{A}^2 \dashrightarrow \mathbb{P}^2$ def.

Def A <u>rational function</u> on X is a rat. map $f: X \dashrightarrow \mathbb{A}^1 = k$.

f is given by regular fcn. $f: U \to k$, $\emptyset \neq U \subseteq X$ open.

$k(X) = \{f : X \dashrightarrow k\}$ field of rat. fcns. on X.

Note: If $(U, f), (V, g) \in k(X)$ then $f+g$, $f-g$, $fg : U \cap V \to k$ rat fcns.

If $f \neq 0$ then $1/f : D(f) \to k$ rat. fcn. on X.

$\therefore k(X)$ is a field.

Example $k(\mathbb{A}^n) = k(x_1, \dots, x_n)$

$k(\mathbb{P}^n) = k(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) = \{ p/q \mid p, q \in k[x_0, \dots, x_n] \text{ forms of same dg} \}$.

Prop X irred variety.

(a) $\emptyset \neq U \subseteq X$ open $\Rightarrow k(X) = k(U)$.

(b) X affine $\Rightarrow k(X) = k[X]_0$ = field of fractions of $k[X]$.

$\therefore k(X)$ <u>easy</u> to compute!

Proof (a) $k(X) \to k(U)$, $(V, f) \mapsto (V \cap U, f|_{V\cap U})$ iso.

(b) Def. $k[X]_0 \to k(X)$, $f/g \mapsto (D(g), f/g)$.

Injective: If $f/g = p/q$ on $D(g) \cap D(q)$ then $fq = pg$ on $D(gq)$

$\Rightarrow fq = pg$ on $X \Rightarrow f/g = p/q \in k[X]_0$.

<u>Surjective</u>: $f : U \to k$ rat. fcn. on X.
$D(g) \subseteq U$ for some $0 \neq g \in k[X]$. Then $f \in k[D(g)] = k[X]_g \subseteq k[X]_0$.
□

<u>Def</u> $(U,f) : X \dashrightarrow Y$ is <u>dominant</u> if $\overline{f(U)} = Y$.
This is indep. of representative (U,f): $\emptyset \neq V \subseteq U$ open $\Rightarrow \overline{f(V)} = Y$. (Exerc.)

Suppose $(U,f) : X \dashrightarrow Y$ dominant and $(V,g) : Y \dashrightarrow Z$ any rat. map.
Can define composition $g \circ f : X \dashrightarrow Z$.
($\overline{f(U)} = Y \Rightarrow f(U) \cap V \neq \emptyset \Rightarrow f^{-1}(V) \subseteq U$ not empty. Take $g \circ f = (f^{-1}(V), g \circ f)$.

<u>Exercise</u> f, g both dominant $\Rightarrow$ $g \circ f$ dominant.

~~Def $f: X \dashrightarrow Y$ is birational if f dominant and $\exists$ dominant $g: Y \dashrightarrow X$ s.t. $f \circ g = id_Y$ and $g \circ f = id_X$.~~
~~X and Y are birationally equivalent (birational), written $X \simeq Y$, if $\exists$ birat. map $X \dashrightarrow Y$.~~

<u>Prop</u> X, Y irred. varieties. 1-1 corresp:
$$\{\text{dominant } \varphi : X \dashrightarrow Y\} \leftrightarrow \{\text{field ext. } k(Y) \to k(X) \text{ preserving } k\}.$$
$$\varphi \mapsto \varphi^* = [h \mapsto h \circ \varphi].$$

<u>Proof</u>
WLOG: X, Y affine.

<u>$\varphi \mapsto \varphi^*$ injective</u>: Assume $\psi : X \dashrightarrow Y$ dominant and $\psi^* = \varphi^*$.
Take $D(h) \subseteq X$ s.t. φ, ψ both def. on $D(h)$.

The diag.
$$\begin{array}{ccc} k(Y) & \xrightarrow{\varphi^* = \psi^*} & k(X) \\ \cup & & \cup \\ k[Y] & \dashrightarrow & k[X]_h \end{array}$$
shows that $\varphi = \psi$ on $D(h) \subseteq X$
$\Rightarrow \varphi = \psi$ as rat. maps.

<u>$\varphi \mapsto \varphi^*$ surjective</u>: Let $\alpha : k(Y) \to k(X)$ homomorphism fixing k.
$k[Y]$ gen by $f_1, \dots, f_n$.
$\alpha(f_i) = g_i / h_i$, $g_i, h_i \in k[X]$.
Set $h = \prod_{i=1}^n h_i$. Then $\alpha(f_i) \in k[X]_h$.
k-alg. hom $\alpha : k[Y] \to k[X]_h$ gives morphism $\varphi : D(h) \to Y$ with $\varphi^* = \alpha$.
□

Def $f: X \dashrightarrow Y$ is <u>birational</u> if f dominant and $\exists$ dom. $g: Y \dashrightarrow X$ s.t. $f \circ g = id_Y$ and $g \circ f = id_X$.

X and Y are <u>birationally equivalent</u> (birational, written $X \approx Y$) if $\exists$ birational $f: X \dashrightarrow Y$.

Examples 1) $\mathbb{A}^2 \approx \mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$.

2) $U, V \subseteq X$ open, X irred. $\Rightarrow U \approx V$.

Thm TFAE: (a) $X \approx Y$ (b) $k(X) \cong k(Y)$ as k-algs.

(c) $\exists$ open $\emptyset \neq U \subseteq X$, $V \subseteq Y$ s.t. $U \cong V$ as vars.

Proof

(a) $\Rightarrow$ (b) clear, first prop.

(b) $\Rightarrow$ (a) 2nd prop.

(a) $\Rightarrow$ (c): Let $(U, f): X \dashrightarrow Y$ and $(V, g): Y \dashrightarrow X$ be inverses.

Set $U_0 = f^{-1}(V) \subseteq U$.

$g \circ f: U_0 \longrightarrow V \longrightarrow X$ morphism of varieties, must be inclusion $U_0 \subseteq X$.

So $g(f(U_0)) \subseteq U_0 \Rightarrow f(U_0) \subseteq g^{-1}(U_0)$.

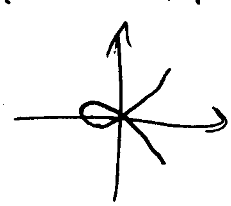
Set $V_0 = g^{-1}(U_0) \subseteq V$. Then $V_0 \cong U_0$.

$\square$

Def An irred. variety is <u>birational</u> if it is birational to some $\mathbb{A}^n$. (one can parametrize a dense open subset.)

Example Any curve $C \subseteq \mathbb{P}^2$ of degree 2 is birational. (HW).

Example $C = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$. is rational.



Def. $\varphi: C \setminus \{(0,0)\} \longrightarrow \mathbb{A}^1$, $\varphi(x,y) = y/x$.

Then $\varphi: C \dashrightarrow \mathbb{A}^1$ is birat. with inverse

$\psi(t) = (t^2 - 1, t^3 - t)$, $\psi: \mathbb{A}^1 \to C$.

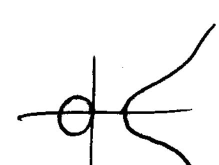
$\mathbb{A}^1 \overset{\psi}{\dashrightarrow} C \overset{\varphi}{\dashrightarrow} \mathbb{A}^1$ is identity.

If $(x,y) \in C$ and $t = y/x$ then $t^2 = \frac{y^2}{x^2} = \frac{x^3 + x^2}{x^2} = x + 1$

$\Rightarrow t^2 - 1 = x$, $y = tx = t^3 - t$.

$\therefore C \dashrightarrow \mathbb{A}^1 \dashrightarrow C$ identity.

Challenge $E = V(y^2 - x^3 + x) \subseteq \mathbb{A}^2$ is <u>not</u> rational.



Big Challenge If C irred. variety and $\exists$ dominant $\mathbb{A}^1 \dashrightarrow C$ then C is rational.

Transcendence degree.

Let $k \subseteq L$ field extension.

- L is algebraic over k if $\forall f \in L$ $\exists$ $f^n + a_1 f^{n-1} + \dots + a_n = 0$, $a_i \in k$.
- A subset $S \subseteq L$ is alg. independent over k if $\forall$ $s_1, \dots, s_n \in S$: $k[x_1, \dots, x_n] \to L$, $x_i \mapsto s_i$ is injective. $(s_i \neq s_j)$.

Def A transcendence basis for L over k is an alg. indep. set $B \subseteq L$ s.t. L is alg. over $k(B)$.

Thm (a) All transcendence bases have same cardinality.
(b) If $S \subseteq \Gamma \subseteq L$ are subsets s.t. S alg. indep. $/k$ and L alg. over $k(\Gamma)$ then $\exists$ tr. basis B for L/k s.t. $S \subseteq B \subseteq \Gamma$.

Proof [redacted] Similar to proof that any vectorspace has a basis.
Easier if L f.g. field ext. of k (Exercise).
See Lang: Algebra.
$\square$

Def $\text{tr.deg}_k(L) = \#$ elts. in tr. basis for L over k.

Def X irred. variety. Set $\dim(X) = \text{tr.deg}\, k(X)$.
Curves = varieties of dim 1.
Surfaces = vars. of dim 2.
n-folds = vars of dim n.

Examples (1) $\dim(\mathbb{A}^n) = \text{tr.deg.}\, k(x_1, \dots, x_n) = n$.
(2) If $\dim(X) = 0$ then $k(X)$ alg. over $k \Rightarrow k(X) = k \Rightarrow X = \text{point}$.

Notation R f.g. domain over k. Write $\text{tr.deg.}\, R = \text{tr.deg}\,(R_0)$.

Principal ideal Theorem (PIT)
R f.g. domain $/k$, $0 \neq f \in R$.
If $P \subseteq R$ min. prime containing f, then $\text{tr.deg}\,(R/P) = \text{tr.deg}(R) - 1$.

Proof: Eisenbud.

Geometric Statement: X irred. var., $0 \neq f \in k[X]$, $Z \subseteq V(f)$ irred. comp. Then $\dim(Z) = \dim(X) - 1$.

Proof
Take $U \subseteq X$ open affine, $U \cap Z \neq \emptyset$.
$Z \cap U = V(P) \subseteq U$, $P \subseteq k[U]$ prime ideal.
Z component of $V(f) \Rightarrow P$ min. prime over (f).
$\therefore \dim Z = \text{tr.deg.}\, R/P = \text{tr.deg.}(R) - 1 = \dim(X) - 1$.
$\square$

$k \subseteq L$ field ext.

Def A transcendense basis for L/k is a subset $B \subseteq L$ s.t. B alg. indep $/k$ and L alg. over $k(B)$.

Def $\text{tr.deg}_k(L) = \text{tr.deg}(L) = \# B$.

Def X irred. variety. Set $\dim(X) = \text{tr.deg. } k(X)$.

curves = varieties of dim. 1
surfaces = —"— dim 2
n-folds = —"— dim n.

Examples (1) $\dim(\mathbb{A}^n) = \text{tr.deg. } k(x_1, \ldots, x_n) = n$. $B = \{x_1, \ldots, x_n\}$.
(2) If $\dim(X) = 0$ then $k(X)$ alg. over $k \Rightarrow k(X) = k. \Rightarrow X = \text{point}$.

Notation R f.g. domain over k. Write $\text{tr.deg.}(R) = \text{tr.deg.}(R_0)$.

Principal Ideal Thm (PIT)
R f.g. domain $/k$, $0 \neq f \in R$. If $P \subseteq R$ min. prime containing f, then $\text{tr.deg.}(R/p) = \text{tr.deg.}(R) - 1$.

Proof: Eisenbud. ($\leq$ easy, $\geq$ harder)

Geometric Statement. X irred. variety, $0 \neq f \in k[X]$, $Z \subseteq V(f)$ irred. comp. Then $\dim(Z) = \dim(X) - 1$.

Proof
Take $U \subseteq X$ open affine, $U \cap Z \neq \emptyset$.
$Z \cap U = V(P) \subseteq U$, $P \subseteq k[U]$ prime ideal.
Z component of $V(f) \Rightarrow P$ min. over $(f) \subseteq k[U]$.
$\therefore \dim(Z) = \text{tr.deg.}(R/p) = \text{tr.deg.}(R) - 1 = \dim(X) - 1$. □

Thm X irred. variety. $\emptyset \neq X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$ max chain of irred. closed subsets. Then $\dim(X) = n$.

Proof WLOG: X affine.
Take $0 \neq f \in I(X_{n-1})$. Then $X_{n-1} \subseteq V(f)$ component $\Rightarrow \dim X_{n-1} = \dim X - 1$.
Induction $\Rightarrow \dim(X_{n-1}) = n-1 \Rightarrow \dim X = n$.
□

Def X any variety. $\dim(X) =$ suppremum of all n s.t. $\exists$ chain of closed irred. $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subseteq X$.

Exercise (a) $X = X_1 \cup \cdots \cup X_n$, X_i closed $\Rightarrow \dim X = \max \dim X_i$
(b) $\dim X \times Y = \dim X + \dim Y$.

Recall: R any ring $\Rightarrow \dim(R) =$ suppremum of n s.t. $\exists$ chain of primes $P_n \subsetneq \cdots \subsetneq P_0 < R$.

Note: X affine $\Rightarrow \dim(X) = \dim k[X]$.

PIT for several equations

X irred. variety, $f_1, \dots, f_r \in k[X]$, $Z \subseteq V(f_1, \dots, f_r)$ any component.
Then $\dim(Z) \geq \dim(X) - r$.

Proof

Enough: $W \subseteq X$ closed and each comp. of W has $\dim \geq d$, then each comp. of $W \cap V(f)$ has $\dim \geq d-1$ for any $f \in k[X]$.

Let $Z \subseteq W$ component.

If $f|_Z = 0$ then $V(f) \cap Z = Z$.

If $f|_Z \neq 0$ then every comp of $V(f) \cap Z$ has $\dim = \dim(Z) - 1 \geq d-1$ (by PIT)

$\therefore$ $W \cap V(f)$ = union of irred. closed sets of $\dim \geq d-1$.

Components of $W \cap V(f)$ are among these sets.

$\square$

Lemma (Prime avoidance)

X affine variety, $Z \subseteq X$ irred. closed subset.
$X_1, \dots, X_m \subseteq X$ irred. closed subsets s.t. $X_i \nsubseteq Z$. $\forall i$.
Then $\exists$ $f \in I(Z)$ s.t. $f \notin I(X_i)$ $\forall i$.

Proof

Induction on m.

$m = 1$: $X_1 \nsubseteq Z \Rightarrow I(Z) \nsubseteq I(X_1) \Rightarrow \exists f \in I(Z) \setminus I(X_1)$.

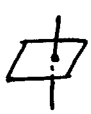
$m \geq 2$: Take $f_i \in I(Z)$ s.t. $f_i \notin I(X_j)$ for $j \neq i$.

If any $f_i \notin I(X_i)$ then done!

If $f_i \in I(X_i)$ $\forall i$ then take $f = f_1 + f_2 f_3 \cdots f_m$.

$\square$

Def X any variety, $Z \subseteq X$ closed irred. subset.
Let $X_1, \dots, X_m$ be the components of X containing Z.
$\operatorname{codim}(Z; X) := \dim(X_1 \cup \dots \cup X_m) - \dim(Z)$.

Example $X =$ . $p \in X$ point. $\operatorname{codim}(p; X) = \begin{cases} 2 & \text{if } p \in \text{plane} \\ 1 & \text{if } p \notin \text{plane} \end{cases}$

Reverse PIT

X affine, $Z \subseteq X$ irred. closed, $c = \operatorname{codim}(Z; X)$.
Then $\exists$ $f_1, \dots, f_c \in k[X]$ s.t. Z is a comp. of $V(f_1, \dots, f_c)$.

Proof

If Z comp. of X then $c = 0$, done.

otherwise: no comp. of X is contained in Z.

Lemma $\Rightarrow$ $\exists$ $f_1 \in I(Z)$ s.t. f_1 does not vanish on any comp. of X.

PIT $\Rightarrow$ $\operatorname{Codim}(Z; V(f_1)) < c$.

Induction on c $\Rightarrow$ $\exists$ $f_2, \dots, f_c \in k[X]$ s.t. Z comp. in $V(f_1, \dots, f_c)$.

$\square$

Resultants K field.

$f(T) = a_n T^n + \dots + a_1 T + a_0$, $\quad g(T) = b_m T^m + \dots + b_1 T + b_0 \quad \in K[T]$.

Q: Do $f(T)$ and $g(T)$ have a common factor?

Set
$$A = \begin{array}{cl} \left[\begin{array}{ccccccc} a_n & a_2 & a_1 & a_0 & & & \\ & & a_n & a_2 & a_1 & a_0 & \\ b_m & b_1 & b_0 & & & & \\ & & b_m & b_1 & b_0 & & \\ & & & & b_m & b_1 & b_0 \end{array}\right] & \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} m \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n \end{array} \end{array}$$

Def $\mathrm{Res}(f,g) = \det(A) \in K$.

Let $\vec{v} = (c_{m-1}, c_0, d_{n-1}, d_1, d_0) \in K^{m+n}$.

Write $\vec{v} \cdot A = (r_{m+n-1}, \dots, r_1, r_0)$

Then $(c_{m-1} T^{m-1} + \dots + c_1 T + c_0) \cdot f(T) + (d_{n-1} T^{n-1} + \dots + d_1 T + d_0) \cdot g(T)$

$$= r_{m+n-1} T^{m+n-1} + \dots + r_1 T + r_0.$$

Prop Suppose $a_n \neq 0$. Then $\mathrm{Res}(f,g) = 0 \iff (f,g) \neq 1 \in K[T]$.

Proof $\mathrm{Res}(f,g) = 0 \iff \exists\, \vec{v} \in K^{n+m} : \vec{v} \cdot A = \vec{0}$

$\iff \exists\, p(T), q(T)$ of deg. at most $m-1$, $n-1$ s.t.

$p(T) \cdot f(T) = q(T) \cdot g(T)$

$\square \quad \iff (f,g) \neq 1$.

If $f(T) = \sum_{i=0}^{n} a_i T^i$, set $f'(T) = \sum_{i=1}^{n} i \cdot a_i \cdot T^{i-1} \in K[T]$.

Notice: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

Cor $f(T) = a_n T^n + \dots + a_1 T + a_0$ has n different roots in $\overline{K} \iff \mathrm{Res}(f, f') \neq 0$.

Proof $f(T) = a_n \prod_{i=1}^{r} (T - \alpha_i)^{d_i}$, $\quad \alpha_i \in \overline{K}$, $\quad \alpha_i \neq \alpha_j$ if $i \neq j$.

$f'(T) = a_n \sum_{i=1}^{r} d_i (T - \alpha_i)^{d_i - 1} \prod_{j \neq i} (T - \alpha_j)^{d_j}$

$\mathrm{Res}(f, f') = 0$

$\Updownarrow$

$(f, f') \neq 1$

$\Updownarrow$

$f'(\alpha_i) = 0$ for some i

$\Updownarrow$

$d_i \geq 2$ for some i.

$\square$

Def $\mathrm{Res}(f, f')$ = discriminant of $f(T)$.

Exercise If $a \neq 0$ then $aT^2 + bT + c$ has discriminant $-a(b^2 - 4ac)$.

Remark If $\mathrm{char}(k)=0$ and $f(T)$ irred. poly, then

$(f(T), f'(T)) = 1 \Rightarrow \mathrm{Res}(f, f') \neq 0$.

Not true in char. p!

$f'(T)$ may be zero. E.g. $(T^p+1)' = 0$.

Remark $\varphi: X \to Y$ morphism. Then $\dim(\overline{\varphi(X)}) \leq \dim(X)$.

Thm $\varphi: X \to Y$ dominant morphism of irred. varieties.

$k(Y) \subseteq k(X)$ finite ext. of deg. d.

Suppose $\mathrm{char}(k)=0$ (or $k(X)/k(Y)$ seperated.)

Then $\exists$ dense open $V \subseteq Y$ s.t. $\# \varphi^{-1}(y) = d \quad \forall y \in V$.

Exercise Find counterexample when $\mathrm{char}(k) = p > 0$.

Proof Let $V \subseteq Y$ and $U \subseteq \varphi^{-1}(V) \subseteq X$ open affines.

$\dim \overline{\varphi(X \setminus U)} \leq \dim(X \setminus U) < \dim(X) = \dim(Y)$.

$\Rightarrow \overline{\varphi(X \setminus U)} \subseteq Y$ proper closed subset.

$\Rightarrow \exists h \in k[V]$ s.t. $D(h) \cap \varphi(X \setminus U) = \emptyset$. $\quad \therefore \varphi^{-1}(D(h)) = D(\varphi^*(h)) \subseteq U$

Replace X with $D(\varphi^*(h))$ and Y with $D(h)$.

WLOG: X, Y both affine.

φ dominant $\Rightarrow k[Y] \subseteq k[X]$.

$k[X]$ gen by $f_1, \ldots, f_n$: $k[Y] \subseteq k[Y][f_1] \subseteq \cdots \subseteq k[Y][f_1, \ldots, f_n] = k[X]$.

Gives $X = X_n \to X_{n-1} \to \cdots \to X_1 \xrightarrow{\psi} Y$. seq. of dom. morphisms.

Induction on n: $\exists$ dense open $U \subseteq X_1$ s.t. all points of U are hit by exactly $d_1 = [k(X) : k(X_1)]$ points of X.

As above: $\overline{\psi(X_1 \setminus U)} \subsetneq Y$ proper closed $\Rightarrow$

$\exists h \in k[Y]$ s.t. $D(h) \cap \psi(X_1 \setminus U) = \emptyset$

$\therefore \psi^{-1}(D(h)) = D(\psi^* h) \subseteq U$.

Enough: $\exists$ open dense $V \subseteq D(h)$ s.t. $\# \psi^{-1}(y) = [k(X_1) : k(Y)] \ \forall y \in V$.

(since $[k(X) : k(Y)] = [k(X) : k(X_1)] \cdot [k(X_1) : k(Y)]$.)

Note: $k[D(\psi^* h)] = k[X_1]_h = k[Y]_h[f_1] = k[D(h)][f_1]$.

Replace X with $D(\psi^* h)$ and Y with $D(h)$.

WLOG $k[X] = k[Y][f_1]$.

Lemma X, Y irred affine, $\varphi: X \to Y$ dominant morphism, $k[X] = k[Y][f]$.
$k(X)/k(Y)$ finite of deg. d.
Then $\exists$ dense open $V \subseteq Y$ s.t. $\# \varphi^{-1}(y) = d \quad \forall y \in V$.

Proof
Let $P(T) = a_d T^d + \dots + a_1 T + a_0 \in k(Y)[T]$ min. poly for $f \in k(X)$.
i.e. $p(f) = 0 \in k(X)$.

WLOG: $a_i \in k[Y] \quad \forall i$.
Replace Y with $D(a_d)$ and X with $D(\varphi^* a_d)$. May assume $a_d = 1$.
Now $k[X] = k[Y][f] \cong k[Y][T]/(P(T))$
$\Rightarrow X \cong V(P) \subseteq Y \times \mathbb{A}^1$ and $\varphi: X \subseteq Y \times \mathbb{A}^1 \xrightarrow{\pi_Y} Y$
If $(y,t) \in Y \times \mathbb{A}^1$ then
$(y,t) \in X \Leftrightarrow P_y(t) = t^d + a_{d-1}(y) t^{d-1} + \dots + a_1(y) t + a_0(y) = 0$
Since $\mathrm{char}(k) = 0$: $\quad \Delta = \mathrm{Res}(P, P') \neq 0 \in k[Y]$.
Notice: $\mathrm{Res}(P_y, P_y') = \Delta(y)$.
So if $y \in D(\Delta) \subseteq Y$ then $P_y(t) = 0$ has exactly
d different solutions in t, i.e. $\# \varphi^{-1}(y) = d$.
$\square$

Exercise $\pi_Y : X \times Y \to Y$ is always an open map.

$f(T),\ g(T) \in K[T]$, K field.

$\mathrm{Res}(f,g) \in K$ polynomium in coefs of f, g.

$\mathrm{Res}(f,g) = 0 \iff (f,g) \neq 1$.

$\mathrm{Res}(f,f') \neq 0 \iff f(T)$ has no multiple roots in $\overline{K}$.

Always true if f irreducible and $\mathrm{char}(K)=0$

Thm $\varphi : X \to Y$ morphism of irred. varieties, s.t. $k(Y) \subseteq k(X)$ finite ext. of deg. d.
Suppose $\mathrm{char}(k) = 0$ (or $k(X)/k(Y)$ separable)
Then $\exists$ dense open $V \subseteq Y$ s.t. $\#\varphi^{-1}(y) = d$ $\forall y \in V$.

Proof

Assume first: X, Y affine and $k[X] = k[Y][f]$ gen. by one elt.
Let $P(T) = a_d T^d + \dots + a_1 T + a_0 \in k(Y)[T]$ min. poly for $f \in k(X)$.
I.e. $P(f) = 0 \in k(X)$.
WLOG: $a_i \in k[Y]$ $\forall i$.
Replace Y with $D(a_d)$ and X with $\varphi^{-1}(D(a_d)) = D(\varphi^* a_d)$.
May assume $a_d = 1$.
Now: $k[X] = k[Y][f] = k[Y][T]/(P(T))$.
$\Rightarrow X \cong V(P) \subseteq Y \times \mathbb{A}^1$ and $\varphi : X \subseteq Y \times \mathbb{A}^1 \xrightarrow{\pi_Y} Y$.
If $(y,t) \in Y \times \mathbb{A}^1$ then:
$$(y,t) \in X \iff P_y(t) = t^d + a_{d-1}(y)t^{d-1} + \dots + a_1(y)t + a_0(y) = 0, \quad P_y(T) \in k[T].$$
Since $\mathrm{char}(k) = 0$: $\Delta = \mathrm{Res}(P, P') \neq 0 \in k[Y]$.
Note: $\mathrm{Res}(P_y, P_y') = \Delta(y)$
So if $y \in D(\Delta) \subseteq Y$ then $P_y(t) = 0$ has exactly d different sols.
$\Rightarrow \#\varphi^{-1}(y) = d$.

General case: X, Y irred. vars, $\varphi : X \to Y$ dominant.
Let $V \subseteq Y$ and $U \subseteq \varphi^{-1}(V)$ open affines.
$\dim \overline{\varphi(X \setminus U)} \leq \dim(X \setminus U) < \dim(X) = \dim(Y)$
$\Rightarrow \overline{\varphi(X \setminus U)} \subsetneq Y$ proper closed subset.
$\Rightarrow \exists\, h \in k[V]$ s.t. $D(h) \cap \varphi(X \setminus U) = \emptyset$
$\therefore \varphi^{-1}(D(h)) = D(\varphi^* h) \subseteq U$.
Replace X with $\varphi^{-1}(D(h))$ and Y with $D(h)$.
WLOG: X, Y affine.

φ dominant $\Rightarrow k[Y] \subseteq k[X]$.

$k[X]$ gen. by $f_1, \dots, f_n$: $\quad k[Y] \subseteq k[Y][f_1] \subseteq \cdots \subseteq k[Y][f_1, \dots, f_n] = k[X]$

Gives $\quad X = X_n \to X_{n-1} \to \cdots \to X_1 \xrightarrow{\psi} Y$ seq. of dominant morphisms.

Induction on n: $\exists$ dense $U \subseteq X_1$ s.t. all points of U are hit by exactly $d_1 = [k(X) : k(X_1)]$ points of X.

As above: $\overline{\psi(X_1 \setminus U)} \subsetneq Y \Rightarrow$

$\exists h \in k[Y]$ s.t. $D(h) \cap \psi(X_1 \setminus U) = \emptyset \Rightarrow \psi^{-1}(D(h)) = D(\psi^* h) \subseteq U$.

$\psi : D(\psi^* h) \to D(h)$, $\quad k[D(\psi^* h)] = k[X]_{\psi^* h} = k[X]_h = k[Y][f_1]_h = k[Y]_h[f_1]$.

First case $\Rightarrow \exists\ \emptyset \neq V \subseteq D(h) \subseteq Y$ open s.t.

$\# \psi^{-1}(y) = [k(X_1) : k(Y)] \quad \forall\, y \in V$.

Since $\psi^{-1}(y) \subseteq U \subseteq X_1$, we get $\# \varphi^{-1}(y) = [k(X_1) : k(Y)] \cdot [k(X) : k(X_1)] = d$, as required.

□

<u>Exercise</u> $\pi_Y : X \times Y \to Y$ is an open map. $\quad U \subseteq X \times Y$ open $\Rightarrow \pi_Y(U) \subseteq Y$ open.

<u>Cor</u> $\varphi : X \to Y$ dominant morphism. Then $\varphi(X) \supseteq$ dense open subset of Y.

<u>Proof</u> WLOG X, Y affine.

Choose $B = \{f_1, \dots, f_n\} \subseteq k[X]$ s.t. B trans. basis for $k(X)/k(Y)$.

$k[Y] \subseteq k[Y][f_1, \dots, f_n] \subseteq X$ gives $\quad X \xrightarrow{\psi} Y \times \mathbb{A}^n \xrightarrow{\pi_Y} Y$, $\quad \varphi = \pi_Y \circ \psi$

Thm $\Rightarrow \exists\ \emptyset \neq U \subseteq Y \times \mathbb{A}^n$ open s.t. $U \subseteq \psi(X)$.

Exercise $\Rightarrow \pi_Y(U) \subseteq Y$ open.

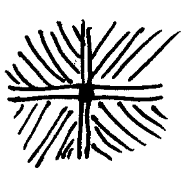
□

<u>Def</u> X variety. $\quad$ (1) $W \subseteq X$ is <u>locally closed</u> if $W =$ open $\cap$ closed.

(2) $W \subseteq X$ is <u>constructible</u> if $W =$ union of finitely many locally closed sets.

<u>Example</u>

$W = D(xy) \cup \{(0,0)\} \subseteq \mathbb{A}^2$ is constructible.



Note: $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$, $\quad \varphi(x, y) = (x^2 y, xy)$.

$W = \varphi(\mathbb{A}^2)$.

<u>Exercise</u> $\varphi : X \to Y$ any morphism of varieties. $\Rightarrow \varphi(X)$ constructible.

<u>Local Rings</u> X irred. variety, $x \in X$.

<u>Def</u> $\mathcal{O}_{X,x} = \{ f \in k(X) \mid f \text{ is defined at } x \}$.

This is a local ring with max. ideal $m_x = \{ f \in \mathcal{O}_{X,x} \mid f(x) = 0 \}$.

<u>Notice</u> If $U \subseteq X$ open, then $k[U] = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq k(X)$.

Let $U \subseteq X$ open affine, $x \in U$.

$M = I(\{x\}) \subseteq k[U]$.

$f \in \mathcal{O}_{X,x} \Rightarrow f$ defined on $D(h) \subseteq U$ for some $h \in k[U] \setminus M$

$\Rightarrow f = g/h^n$, $g \in k[U]$, $n \geq 0$.

$\therefore \mathcal{O}_{X,x} = \{ g/h : g, h \in k[U], h(x) \neq 0 \} = k[U]_M \subseteq k(X)$.

<u>Remark</u> X not. irred. $\Rightarrow \mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U)$.

If $U \subseteq X$ affine, $x \in U$, still have $\mathcal{O}_{X,x} = k[U]_M$.

<u>Note:</u> X irred. $\Rightarrow \dim(X) = \dim \mathcal{O}_{X,x} \quad \forall x \in X$.

X any variety $\Rightarrow \dim(X) = \max_{x \in X} \dim \mathcal{O}_{X,x}$.

<u>Def</u> (R, m) local ring. $F = R/m$ field. m/m^2 F-vector space.

R is a <u>regular local ring</u> $\Leftrightarrow \dim(R) = \dim_F(m/m^2)$.

<u>Exercise</u> R Noetherian local ring. Then

$\dim_F(m/m^2) \underset{NAK}{=}$ min. # generators for $m \underset{PIT}{\geq} \dim(R)$.

<u>Def</u> X variety, $x \in X$.

1) X is <u>non-singular</u> at x if $\mathcal{O}_{X,x}$ is regular local.

2) X is <u>singular</u> at x if not.

3) X is <u>non-singular</u> if all points are non-sing.

<u>Note:</u> S commutative ring, $M \subseteq S$ max. ideal.

$R = S_M$, $m = MS_M \subseteq R$.

Then $S/M \cong R/m$ and $M/M^2 \cong m/m^2$.

<u>Example</u> $C = V(f) \subseteq \mathbb{A}^2$ curve, $P = (0,0) \in C$.

$f = ax + by +$ higher terms. $k[C] = k[x,y]/(f)$.

$m_P/m_P^2 = (x,y)/(x^2, xy, y^2, f) = (x,y)/(x^2, xy, y^2, ax+by)$.

C non-sing. at $P = (0,0) \Leftrightarrow \dim_k(m_P/m_P^2) = 1 \Leftrightarrow ax + by \neq 0 \in k[x,y]$.

Note: $ax + by \neq 0 \Rightarrow V(f)$ looks like $V(ax+by)$ close to P.

__Exercise__ Find all singular points of $V(y^2-x^3-x^2)$, $V(y^2-x^3) \subseteq \mathbb{A}^2$.

Let $X \subseteq \mathbb{A}^n$ closed, affine variety.

$I = I(X) = (f_1, .., f_t) \subseteq S = k[x_1, ..., x_n]$.

__Idea:__ X non-singular at $p \iff X$ has tangent plane at p.

__Def__ $J_p = \left[\frac{\partial f_i}{\partial x_j}(p) \right]$ $t \times n$ __Jacobian matrix.__

__Note:__ If $\vec{v} \in k^n$ then $J_p \cdot \vec{v}$ = directional derivative of $(f_1, .., f_t)$ along $\vec{v}$.

$\mathrm{Ker}(J_p) = \{ \vec{v} \in k^n \mid J_p \cdot \vec{v} = \vec{0} \}$ candidate for tangent space.

__Lemma__ $p \in X \subseteq \mathbb{A}^n \Rightarrow \mathrm{rank}(J_p) + \dim_k (\mathfrak{m}_p/\mathfrak{m}_p^2) = n$.

__Proof__ Set $M = I(\{p\}) \subseteq S$.

__Def__ $d : M \longrightarrow k^n$, $d(f) = \left(\frac{\partial f}{\partial x_1}(p), .., \frac{\partial f}{\partial x_n}(p) \right)$

Surjective: $d(x_i - p_i) = e_i$

Note: $d(f \cdot g) = f(p) \cdot d(g) + g(p) \cdot d(f)$ for $f, g \in S$.

$\Rightarrow d(M^2) = 0$.

$\therefore$ $d : M/M^2 \xrightarrow{\cong} k^n$.

$d(f_i)$ = row i of J_p.

$$d\left(\sum_{i=1}^t g_i f_i \right) = \sum g_i(p)\, d(f_i)$$

$\Rightarrow d(I)$ = row span of J_p in k^n.

$d : I+M^2/M^2 \xrightarrow{\cong}$ row span of J_p.

$\therefore$ $\mathrm{rank}(J_p) + \dim_k (M/I+M^2) = \dim_k (I+M^2/M^2) + \dim (M/M+I^2) = n$

Finally: $\mathcal{O}_{X,p} = (S/I)_M$,

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = (M/I)/(M/I)^2$$

$$= M/I+M^2$$

$\square$

X variety, $x \in X$.

$\mathcal{O}_{X,x} = \lim\limits_{U \ni x} \mathcal{O}_X(U) = k[X]_{I(x)}$ if X affine

$= \{ f \in k(X) \mid f \text{ defined at } x \}$ if X irreducible.

$m_x \subseteq \mathcal{O}_{X,x}$ max ideal.

Def X <u>non-singular</u> at x if $\mathcal{O}_{X,x}$ <u>regular local ring</u>,

i.e. m_x gen by $\dim \mathcal{O}_{X,x}$ elements $\Leftrightarrow \dim_k (m_x/m_x^2) = \dim \mathcal{O}_{X,x}$.

Exercise S commutative ring, $M \subseteq S$ max ideal.

$R = S_M$, $m = MS_M \subseteq R$.

Then $S/M \cong R/m$ and $M/M^2 \cong m/m^2$.

Example $C = V(f) \subseteq \mathbb{A}^2$ curve. $P = (0,0) \in C$

$f = ax + by +$ higher terms. $k[C] = k[x,y]/(f)$.

$M = I(P) = (\bar{x}, \bar{y}) \subseteq k[C]$. $M = (x,y)/(f)$.

$m_P/m_P^2 = M/M^2 = (x,y)/(x^2, xy, y^2, f) = (x,y)/(x^2, xy, y^2, ax+by)$

C non-sing at $P=(0,0) \Leftrightarrow \dim_k(m_P/m_P^2) = 1 \Leftrightarrow ax+by \neq 0 \in k[x,y]$.

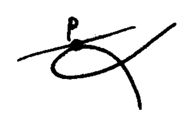
Note: $ax+by \neq 0 \Rightarrow V(f)$ looks like $V(ax+by)$ <u>close to P</u>.

Exercise Find all singular points of $V(y^2 - x^3 - x^2)$ and $V(y^2 - x^3)$.

Let $X \subseteq \mathbb{A}^n$ closed affine variety.

$I = I(X) = (f_1, \dots, f_t) \subseteq S = k[x_1, \dots, x_n]$.

Idea: X non-sing at $p \Leftrightarrow X$ has tangent plane at p.



Def $J_p = \left[\frac{\partial f_i}{\partial x_j}(p) \right]$ $t \times n$ <u>Jacobi matrix</u>.

Note: If $\vec{v} \in k^n$ then $J_p \cdot \vec{v}$ = directional derivative of $(f_1, \dots, f_t)$ at p along $\vec{v}$.

$\mathrm{Ker}(J_p) = \{ \vec{v} \in k^n \mid J_p \cdot \vec{v} = \vec{0} \}$ candidate for tangent space.

Lemma $p \in X \subseteq \mathbb{A}^n \Rightarrow \mathrm{rank}(J_p) + \dim_k(m_p/m_p^2) = n$.

Proof Set $M = I(\{p\}) \subseteq S$.

Def. $d : M \longrightarrow k^n$, $d(f) = \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right)$

Surjective: $d(x_i - p_i) = e_i$

Note: $d(f \cdot g) = f(p) \cdot d(g) + g(p) \cdot d(f)$ for $f, g \in S$.

$\Rightarrow d(M^2) = 0$.

$\therefore d : M/M^2 \xrightarrow{\cong} k^n$ iso.

$d(f_i) =$ row i of J_p.

$d\left(\sum_{i=1}^{t} g_i \cdot f_i\right) = \sum g_i(p)\cdot d(f_i) \Rightarrow d(I) =$ row span of J_p in k^n.

$\therefore\ d: I+M^2/M^2 \xrightarrow{\cong}$ row span of J_p

$\Rightarrow \operatorname{rank}(J_p) + \dim_k(M/I+M^2) = \dim_k(M/M^2) = n.$

Finally: $\mathcal{O}_{X,p} = (S/I)_M$

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = (M/I)/(M/I)^2 = M/I+M^2.$$

□

Thm Let $p \in X \subseteq \mathbb{A}^n$. Then $\operatorname{rank}(J_p) \leq n - \dim \mathcal{O}_{X,p}$.

$\operatorname{rank} J_p = n - \dim \mathcal{O}_{X,p} \iff \mathcal{O}_{X,p}$ regular local.

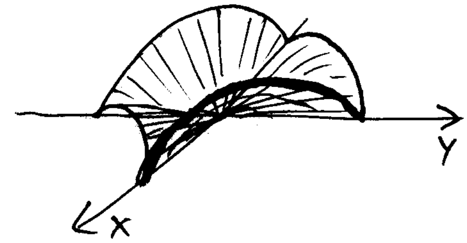
Proof

$\operatorname{rank} J_p = n - \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) \leq n - \dim \mathcal{O}_{X,p}$

Equality $\iff \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = \dim \mathcal{O}_{X,p}$.

□

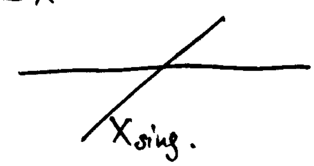
Example $X = V(z^3 - x^2y^2) \subseteq \mathbb{A}^3$, $\operatorname{char}(k) \neq 2,3$.



$J = [-2xy^2, -2x^2y, 3z^2]$

$p \in X$ non-singular pt $\iff \operatorname{rank} J_p = 3 - \dim(X) = 1$

$X_{sing} = V(z^3 - x^2y^2, 2xy^2, 2x^2y, 3z^2) = V(xy, z)$.



Exercise X variety, $p \in X$. $\mathcal{O}_{X,p}$ domain $\iff p \in$ only one component.

Thm Any Noetherian regular local ring is a domain. (In fact a UFD.)

Consequence points in intersection of two components are singular.

Prop $X_{sing} \subseteq X$ is closed.

Proof

$X = X_1 \cup \dots \cup X_r$ components, $X_{sing} = \left(\bigcup_{i=1}^{r} (X_i)_{sing}\right) \cup \left(\bigcup_{i \neq j} X_i \cap X_j\right)$.

WLOG $X \subseteq \mathbb{A}^n$ irred. affine variety.

$I(X) = (f_1, \dots, f_t) \subseteq k[x_1, \dots, x_n]$.

$p \in X_{sing} \iff p \in X$ and $\operatorname{rank} J_p < n - \dim \mathcal{O}_{X,p} = n - \dim(X)$.

Let $m_1, \dots, m_N \in k[x_1, \dots, x_n]$ minors of size $n - \dim(X)$ in $J = \left[\frac{\partial f_i}{\partial x_j}\right]$.

$X_{sing} = V(f_1, \dots, f_t, m_1, \dots, m_N) \subseteq \mathbb{A}^n$.

□

Fact: $X_{sing} \neq X$. Reason: $X \approx$ hypersurface in $\mathbb{P}^n$ + Exercise.

<u>Lemma</u> $p \in X = V(g_1, \dots, g_r) \subseteq \mathbb{A}^n$.

If $\operatorname{rank} J_p(g_1, \dots, g_r) = r$ then $\mathcal{O}_{X,p}$ regular local of dim $n-r$.

<u>Proof</u>

PIT $\Rightarrow$ $\dim \mathcal{O}_{X,p} \geq n-r$.

$I(X) = (f_1, \dots, f_t) \supseteq (g_1, \dots, g_r) \Rightarrow$ row span $J_p(f_1, \dots, f_t) \supseteq$ row span $J_p(g_1, \dots, g_r)$.

$r = \operatorname{rank}(J_p(g_1, \dots, g_r)) \leq \operatorname{rank} J_p(f_1, \dots, f_t) \leq n - \dim \mathcal{O}_{X,p} \leq r$.

$\square$

<u>Implicit Function Thm</u>

$f_1, \dots, f_c$ holomorphic fcns def. in classical nbhd. of $p \in \mathbb{C}^n$.

Suppose $\det\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \leq i,j \leq c} \neq 0$.

Then $\exists$ holomorphic fcns $w_1, \dots, w_c$ on a classical open subset of $\mathbb{C}^{n-c}$ and classical open $V \subseteq \mathbb{C}^n$, $p \in V$ s.t.

$\forall z \in V: \quad f_1(z) = \dots = f_c(z) = 0 \iff z_i = w_i(z_{c+1}, \dots, z_n) \quad \forall\ 1 \leq i \leq c$.

<u>Thm</u> $X \subseteq \mathbb{C}^n$ complex affine variety. $p \in X$ non-sing point.

Then a classical ~~[illegible]~~ open nbhd. of p in X is holomorphic to a classical open subset of $\mathbb{C}^d$, ~~[illegible]~~. $d = \dim(\mathcal{O}_{X,p})$.

<u>Proof</u>

WLOG: X irreducible.

$I(X) = (f_1, \dots, f_t)$. Then $\operatorname{rank}(J_p(f_1, \dots, f_t)) = n-d =: c$.

WLOG: $\det\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \leq i,j \leq c} \neq 0$.

Set $Y = V(f_1, \dots, f_c) \subseteq \mathbb{A}^n$.

$\operatorname{rank} J_p(f_1, \dots, f_c) = c \Rightarrow \mathcal{O}_{Y,p}$ regular local of dim d.

Now: only one component of Y contains p.

$p \in X \subseteq Y$, X irred. of dim d.

$\therefore$ $X =$ component of Y containing p.

$\exists\ U \subseteq \mathbb{A}^n$ open s.t. $p \in U$ and $X \cap U = V(f_1, \dots, f_c) \cap U$.

Use IFT: Let $V \subseteq U$ and $w_1, \dots, w_c$ be as in IFT.

<u>Def</u> $\pi: \mathbb{C}^n \longrightarrow \mathbb{C}^d$, $\pi(z) = (z_{c+1}, \dots, z_n)$.

Then $\pi: X \cap V \xrightarrow{\sim} \pi(X \cap V) \subseteq \mathbb{C}^d$ holomorphism.

Inverse map: $\pi^{-1}(z_{c+1}, \dots, z_n) = (w_1, \dots, w_c, z_{c+1}, \dots, c_n)$.

$\square$

<u>Cor</u> ~~[illegible]~~

Every non-sing complex variety is a complex manifold.

Lemma X irreducible var., $x, y \in X$. If $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ then $x = y$.

Proof Take open affines $x \in U \subseteq X$, $y \in V \subseteq X$.

X separated $\Rightarrow$ $U \cap V$ affine and $k[U] \otimes k[V] \twoheadrightarrow k[U \cap V]$.

$k[U] \subseteq \mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ and $k[V] \subseteq \mathcal{O}_{X,y}$ $\Rightarrow$ $k[U \cap V] \subseteq \mathcal{O}_{X,y}$.

$k[V] \subseteq k[U \cap V] \subseteq \mathcal{O}_{X,y}$ shows:

$k[U \cap V] \cap m_y$ max. ideal in $k[U \cap V]$.

$\Leftrightarrow$ point in $U \cap V$ mapping to y under $U \cap V \hookrightarrow V$.

$\therefore$ $y \in U$.

$A = k[U]$, $x \leftrightarrow P \subseteq A$, $y \leftrightarrow Q \subseteq A$ max. ideals, $A_P \subseteq A_Q$.

$QA_Q \cap A_P \subseteq PA_P$ $\Rightarrow$ $Q = QA_Q \cap A \subseteq PA_P \cap A = P$.

$\therefore$ $Q = P$ since Q max. ideal.

$\square$

Say: Have used that points on affine varieties $\leftrightarrow$ subsets of affine coord rings.
Now points on arbitrary irred. var. $\leftrightarrow$ subsets of function field.

Non-singular curves

(R, m) Noetherian regular local ring of dimension one.

$m = (t)$, t = uniformizing parameter.

$K = R_0$ = field of fractions.

OBS: Any $f \in K^*$ can be written as $f = ut^n$, $u \in R \setminus m$ unit.

WLOG $f \in R \setminus \{0\}$.

If false then choose $f \in R \setminus \{0\}$ not of this form, with $(f) \subseteq R$ maximal.

f not unit $\Rightarrow$ $f \in m$ $\Rightarrow$ $f = tg$.

$(f) \subsetneq (g)$. (If $(f) = (g)$ then $g = hf \Rightarrow f = thf \Rightarrow th = 1$.)

So $g = ut^n \Rightarrow f = ut^{n+1}$!

Check: If $f = ut^n$ then $u \in R \setminus m$ and $n \in \mathbb{Z}$ are unique.

Def

$$v : K^* \longrightarrow \mathbb{Z}$$
$$v(f) = n \quad \text{if } f = ut^n, \ u \in R \setminus m.$$

Rules

$$v(fg) = v(f) + v(g)$$
$$v(f+g) \geq \min(v(f), v(g)).$$

Example: $R = k[t]_{(t)}$. $m = t \cdot R$.

Alg. Geo. 10/16/2006

X affine variety : points of $X \leftrightarrow$ max ideals in $k[X]$.
X irred. variety : $p \in X \leftrightarrow$ local ring $\mathcal{O}_{X,p} = \{f \in k(X) : f(p) \text{ defined}\} \subseteq k$

Lemma X irred. variety, $x, y \in X$. If $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ then $x = y$.

Proof Take open affines $x \in U \subseteq X$, $y \in V \subseteq Y$.
X separated $\Rightarrow$ $U \cap V$ affine and $k[U] \otimes k[V] \to k[U \cap V]$
$k[U] \subseteq \mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ and $k[V] \subseteq \mathcal{O}_{X,y}$ $\Rightarrow$ $k[U \cap V] \subseteq \mathcal{O}_{X,y}$.
$k[V] \subseteq k[U \cap V] \subseteq \mathcal{O}_{X,y}$ shows that $y \in U \cap V$:
$k[U \cap V] \cap m_y$ max. ideal in $k[U \cap V]$ $\leftrightarrow$ point in $U \cap V$ mapping to y under $U \cap V \hookrightarrow V$.
$\therefore$ $x, y \in U \subseteq X$, U affine.
If $x \neq y$ then take $f \in k[U]$ s.t. $f(x) \neq 0$, $f(y) = 0$.
Then $1/f \in \mathcal{O}_{X,x}$ but $1/f \notin \mathcal{O}_{X,y}$.
□

Non-Singular curves.

Def A curve is an irred. variety of dim 1.

Example $C = \mathbb{A}^1$, $k[C] = k[t]$, $k(C) = k(t)$.
Let $0 \neq f \in k(C)$. What is order of vanishing of f at $0 \in C$?
$f = p/q$, $p, q \in k[t]$.
Write $p = p_0 \cdot t^n$, $q = q_0 t^m$, $p_0(0) \neq 0$, $q_0(0) \neq 0$.
$f = (p_0/q_0)\, t^{n-m}$; $v(f) = n - m$ = order of vanishing of f at 0.
Note: $\mathcal{O}_{C,0} = k[t]_{(0)}$, $m_0 = (t) \subseteq \mathcal{O}_{C,0}$ max. ideal.
$f = u\, t^{n-m}$, $u \in \mathcal{O}_{C,0} \setminus m_0$ unit in $\mathcal{O}_{C,0}$.

Def A discrete valuation ring (DVR) is ~~a~~ Noetherian regular local ring of dim 1.
E.g. local ring of curve at non-sing. point.

Let (R, m) be a DVR. $m = (t)$. t = "uniformizing parameter".
Set $K = R_0$ = field of fractions.

Claim Any $f \in K^*$ can be written $f = u t^n$, $u \in R \setminus m$ unit.
WLOG: $f \in R \setminus \{0\}$.
If false, let $f \in R \setminus \{0\}$ be counterexample with (f) maximal.

f not unit $\Rightarrow f \in \mathfrak{m}_* \Rightarrow f = tg$, $g \in R$.

$(f) \subsetneq (g)$: If $(f) = (g)$ then $g = hf \Rightarrow f = thf \Rightarrow th = 1$. ↯

So $g = ut^n \Rightarrow f = ut^{n+1}$.

Check: If $f = ut^n$ then u, n are unique.

<u>Def</u> $v : K^* \longrightarrow \mathbb{Z}$ valuation map, $v(f) = n$ if $f = ut^n$, $u \in R \setminus \mathfrak{m}$

<u>Note</u>: $R = \{ f \in K^* \mid v(f) \geq 0 \} \cup \{0\}$

$\mathfrak{m} = \{ f \in K^* \mid v(f) > 0 \} \cup \{0\}$.

<u>Rules</u>: $v(fg) = v(f) + v(g)$.

$v(f+g) \geq \min(v(f), v(g))$ if $f, g, f+g \in K^*$.

($f = ut^n$, $g = vt^m$, $n \leq m \Rightarrow f+g = (u + vt^{m-n})t^n = u't^r t^n$.)

<u>Example</u> C curve, $P \in C$ non-sing. point.

$R = \mathcal{O}_{C,P}$ DVR with $R_0 = k(C)$.

$v_P : k(C)^* \longrightarrow \mathbb{Z}$ valuation ring given by $\mathcal{O}_{C,P}$.

$f \in k(C)^*$: $v_P(f)$ = order of vanishing of f at P.

$v_P(f) > 0 \iff f \in \mathfrak{m}_P \iff f(P) = 0$

$v_P(f) = 0 \iff f(P) \neq 0$.

$v_P(f) < 0 \iff f$ not defined at P (f has <u>pole</u> at P).

<u>Lemma</u> R DVR, $K = R_0$. S any ring s.t. $R \subseteq S \subseteq K$.
Then $S = R$ or $S = K$.
<u>Proof</u> $\mathfrak{m}_R = (t)$. If $S \neq R$ then take $h \in S \setminus R$. $h = ut^n$, $n < 0$.
$S \supseteq R[h] = K$ ∎

<u>Recall</u> A domain R is <u>integrally closed</u> if
$\forall f \in R_0$: f integral over $R \iff f \in R$.

<u>Thm</u> R Noetherian local domain of dim one.
Then R is regular (DVR) $\iff$ R is integrally closed.

<u>Exercise</u>
Prove "$\Rightarrow$".

<u>Prop</u> A domain A is integrally closed $\iff$ A_P int. closed $\forall P \subseteq A$ max. ideal.

<u>Exercise</u>: $\Rightarrow$ is easy.

<u>Note</u>: If $A = k[X]$, X affine, then $A = \bigcap A_P \subseteq k(X)$, so $\Leftarrow$ easy.

<u>Def</u> A <u>Dedekind domain</u> is an integrally closed Noeth. domain of dim 1.

<u>Note</u>: If X irred. affine variety, then

X non-sing. curve $\iff$ $k[X]$ Dedekind domain.

$\mathcal{O}_{X,P}$ DVR $\forall P \in X$ $\iff$ $k[X]_{\mathfrak{m}_P}$ int. closed $\forall \mathfrak{m}_P$ $\iff$ $k[X]$ int. closed.

Finiteness of integral closure.

R f.g. domain over k. $K = R_0$.

$K \subseteq L$ finite field extension.

$\overline{R}$ = integral closure of R in L $= \{ f \in L : f \text{ integral over } R \}$.

<u>Fact</u>: $\overline{R}$ integrally closed domain with $(\overline{R})_0 = L$.

$(\overline{R})_0 = L$: $f \in L \Rightarrow f^n + a_1 f^{n-1} + \dots + a_n = 0$, $a_i \in K$.

Take $b \in R$ s.t. $b a_i \in R$ $\forall i$.

$(bf)^n + ba_1 (bf)^{n-1} + \dots + b^n a_n = 0 \Rightarrow bf \in \overline{R} \Rightarrow f \in (\overline{R})_0$.

<u>Theorem</u> $\overline{R}$ is a finitely generated R-module.

In particular: $\overline{R}$ = f.g. k-algebra.

<u>Def</u> $k \subseteq K$ field extension. A DVR of K/k is a subring $R \subseteq K$ s.t.

(i) R is a DVR (ii) $R_0 = K$ (iii) $k \subseteq R$.

E.g. $\mathcal{O}_{C,P}$, $P \in C$ non-sing pt.

Let K = function field of dim. one over k,

i.e. $k \subseteq K$ finitely generated field extension of tr. deg. 1.

Q: $\exists$ non-sing curve C s.t. $K = k(C)$?

<u>Key Construction</u>

Let $f \in K \setminus k$. Then $k(f) \subseteq K$ finite field ext.

($k(f) \subseteq K$ alg. since $\{f\}$ transc. basis for K.)

Set $B = \overline{k[f]} \subseteq K$

Finiteness of int. closure $\Rightarrow$ B f.g. k-algebra.

$\therefore$ B Dedekind domain with $B_0 = K$.

<u>Prop</u> $X = \operatorname{Spec}(B)$ non-sing curve.

(1) points in X $\leftrightarrow$ DVRs R of K/k s.t. $f \in R$.

$P \longmapsto \mathcal{O}_{X,P}$

(2) points in $V(f)$ $\leftrightarrow$ DVRs R of K/k s.t. $f \in \mathfrak{m}_R$.

<u>Proof</u> $X \longrightarrow \{\text{DVRs } R \ni f\}$ well defined, injective.

Let R be any DVR of K/k s.t. $f \in R$.

$k[f] \subseteq R \Rightarrow B \subseteq R$.

Set $M = B \cap \mathfrak{m}_R \subseteq B$ prime ideal.

Then $B_M \subseteq R$. In particular $M \neq 0$, so B_M DVR of K/k. Lemma $\Rightarrow B_M = R$. $\square$

4

<u>Cor 1</u> Every DVR R of K/k is the local ring of a non-sing curve at some point. In particular $R/m_R \cong k$.

<u>Proof</u> Let $f \in R \setminus k$, $B = \overline{k[f]} \subseteq K$. Then $R = B_M$ for some $M \in \text{Specm}(B)$

I.e. $R = \mathcal{O}_{X,M}$, $X = \text{Spec-m}(B)$.

□

<u>Cor 2</u> Given $f \in K^*$ there exist finitely many DVRs R of K/k such that $f \in m_R$. Also finitely many s.t. $f \notin R$.

<u>Proof</u>
WLOG $f \notin k$. Set $B = \overline{k[f]}$.

DVRs R of K/k with $f \in m_R$ $\Leftrightarrow$ points of $V(f) \subseteq \text{Spec}^m(B)$.

PIT $\Rightarrow$ $\dim V(f) = 0$ $\Rightarrow$ $V(f)$ finite set.

2nd half: Note that $f \notin R \Leftrightarrow 1/f \in m_R$.

□

<u>Def</u> $C_K =$ ~~[struck out]~~. $\{\text{DVRs of } K/k\}$.

Elements of C_K are called <u>points</u>.

The DVR given by $P \in C_K$ is R_P, max. ideal m_P.

<u>Topology</u>: Closed subsets of C_K are finite sets + all of C_K.

Let $f \in K$ and $P \in C_K$. If $f \in R_P$ then set

$f(P) =$ image of f by $R_P \longrightarrow R_P/m_P \cong k$. I.e. $f(P) \equiv f \pmod{m_P}$.

<u>Regular functions</u>: If $\emptyset \neq U \subseteq C_K$ open, we set $k[U] = \bigcap_{P \in U} R_P \subseteq K$.

$\therefore$ C_K = space with functions.

<u>Note</u>: If $U \subseteq C_K$ open and $f \in k[U]$ then

$D(f) = \{P \in U \mid f(P) \neq 0\} = \{P \in U \mid f \notin m_P\}$ is open. (Cor 2).

<u>Example</u> $K = k(t)$. DVRs of $k(t)/k$ are:

$k[t]_{(t-a)}$ for $a \in k$ and $k[t^{-1}]_{(t^{-1})}$.

$C_K \leftrightarrow \mathbb{P}^1$: $k[t]_{(t-a)} \leftrightarrow a \in \mathbb{A}^1$, $k[t^{-1}]_{(t^{-1})} \leftrightarrow \infty \in \mathbb{P}^1$.

Note: If $f \in k[t]_{(t-a)}$ then $f(t) \equiv f(a) \pmod{(t-a)}$

$\Rightarrow f(k[t]_{(t-a)}) = f(a)$.

Thm R Noetherian local domain of dim 1. DVR $\Leftrightarrow$ integrally closed.

Thm R f.g. domain over k. $K = R_0$, $K \subseteq L$ finite ext. $\overline{R} \subseteq L$ f.g. R-module.

Key construction

K = function field of dim 1 over k. (f.g. ext + tr.deg $= 1$)

Let $f \in K \setminus k$. Then $k(f) \subseteq K$ finite extension. ($\{f\}$ tr. basis.)

Set $B = \overline{k[f]} \subseteq K$. Thm $\Rightarrow$ B f.g. k-algebra.

$\therefore$ B f.g. Dedekind Domain over k, with $B_0 = K$.

Prop $X = \mathrm{Specm}(B)$ is a non-sing. curve.

(a) points in X $\Leftrightarrow$ DVRs R of K/k s.t. $f \in R$

(b) points in $V(f)$ $\leftrightarrow$ DVRs R of K/k s.t. $f \in \mathfrak{m}_R$.

$$P \longmapsto \mathcal{O}_{X,P}$$

Proof

$X \to \{\text{DVRs } R \ni f\}$ well defined, injective.

Let R be any DVR of K/k s.t. $f \in R$

$k[f] \subseteq R \Rightarrow B \subseteq R$.

Set $\mathfrak{M} = B \cap \mathfrak{m}_R \subseteq B$ prime ideal.

Then $B_{\mathfrak{M}} \subseteq R$. $\Rightarrow$ $\mathfrak{M} \neq 0$ $\Rightarrow$ $B_{\mathfrak{M}}$ DVR of K/k.

Lemma $\Rightarrow$ $B_{\mathfrak{M}} = R$ or ~~$R = K$~~

$\therefore$ $B_{\mathfrak{M}} = R$.

$\mathfrak{M} \in V(f) \Leftrightarrow f \in \mathfrak{M} \Leftrightarrow f \in \mathfrak{M} B_{\mathfrak{M}} = \mathfrak{m}$.

□

Cor 1 Every DVR R of K/k is the local ring of a non-sing curve @ point.

In particular $R/\mathfrak{m}_R \cong k$.

Proof

Let $f \in R \setminus k$, $B = \overline{k[f]} \subseteq K$.

Then $R = B_p$ for some $p \in \mathrm{Specm}(B)$.

I.e. $R = \mathcal{O}_{X,p}$, $X = \mathrm{Specm}(B)$.

□

Cor 2 Given $f \in K^*$, there are finitely many DVRs R of K/k s.t. $f \in \mathfrak{m}_R$.

Also finitely many s.t. $f \notin R$.

Proof

WLOG $f \notin k$. Set $B = \overline{k[f]}$. $\{R : f \in \mathfrak{m}_R\} \leftrightarrow V(f) \subseteq \mathrm{Specm}(B)$.

PIT $\Rightarrow$ $\dim V(f) = 0$ $\Rightarrow$ $V(f)$ finite set.

2nd half: Note $f \notin R \Leftrightarrow 1/f \in \mathfrak{m}_R$. □

Def $C_K = \{ \text{DVRs of } K/k \}$

Elements of C_K are called "points" P.

The DVR given by $P \in C_K$ is R_P, max ideal m_P.

Topology: Closed subsets of C_K are finite sets + all of C_K.

Let $f \in K$ and $P \in C_K$. If $f \in R_P$ then set

$f(P)$ = image of f by $R_P \to R_P/m_P = k$. (I.e. $f(P) \equiv f \mod m_P$.)

Regular fcns: If $\emptyset \neq U \subseteq C_K$ open, set $k[U] = \bigcap_{P \in U} R_P \subseteq K$.

$\therefore$ C_K space with functions.

Note If $U \subseteq C_K$ open and $f \in k[U]$, then

$D(f) = \{ P \in U : f(P) \neq 0 \} = \{ P \in U : f \notin m_P \}$ is open (Cor 2)

Example $K = k(t)$. DVRs of $k(t)/k$ are:

$k[t]_{(t-a)}$ for $a \in k$ and $k[t^{-1}]_{(t^{-1})}$.

$C_K \leftrightarrow \mathbb{P}^1$: $k[t]_{(t-a)} \leftrightarrow a \in \mathbb{A}^1$, $k[t^{-1}]_{(t^{-1})} \leftrightarrow \infty \in \mathbb{P}^1$.

Note: If $f \in k[t]_{(t-a)}$ then $f(t) \equiv f(a) \pmod{(t-a)}$.

$\Rightarrow f(k[t]_{(t-a)}) = f(a)$.

Thm C_K is a non-singular curve with $k(C_K) = K$.

Proof Take any $f \in K \setminus k$, set $B = \overline{k[f]} \subseteq K$,

$U = \{ P \in C_K : f \in R_P \}$. Then $U \subseteq C_K$ is open.

Def $\varphi : \operatorname{Specm}(B) \longrightarrow U$, $M \mapsto B_M$.

Prop $\Rightarrow$ φ bijective.

proper closed sets = finite sets $\Rightarrow$ φ homeomorphism.

For $V \subseteq \operatorname{Spec}(B)$ open we have

$$k[V] = \bigcap_{M \in V} B_M = \bigcap_{P \in \varphi(V)} R_P = k[\varphi(V)].$$

$\therefore$ $\varphi : \operatorname{Spec}(B) \cong U$ isomorphism.

Note: If $P \in C_K$ then $f \in R_P$ or $f^{-1} \in R_P$

$\Rightarrow C_K = \operatorname{Specm}(\overline{k[f]}) \cup \operatorname{Specm}(\overline{k[f^{-1}]})$ open affine cover.

$\therefore$ C_K prevariety.

<u>Separated</u>: If $P, Q \in C_K$, show $\exists$ open affine $U \subseteq C_K$: $P, Q \in U$.
Take any $f \in R_P \setminus m_P$ s.t. $f \notin k$. Then $f, f^{-1} \in R_P$.
If $f \in R_Q$ then $P, Q \in \operatorname{Specm}(\overline{k[f]})$.
Otherwise $P, Q \in \operatorname{Specm}(\overline{k[f^{-1}]})$.

<u>C_K irreducible</u>: C_K is infinite and proper closed subsets are finite.
□

<u>Prop</u> C irred. curve. $P \in C$ non-sing point. Y projective variety.
Any morphism $\varphi : C \setminus \{P\} \longrightarrow Y$ can be uniquely extended to morphism $\varphi : C \longrightarrow Y$.

<u>Note</u>: No points are "missing" from Y!

<u>Proof</u> $Y \subseteq \mathbb{P}^n$ closed subset. Enough to construct $\varphi : C \longrightarrow \mathbb{P}^n$.
WLOG: $\varphi(C \setminus \{P\}) \nsubseteq V_+(x_i)$ for all i.
Set $U = D_+(x_0 x_1 \cdots x_n) \subseteq \mathbb{P}^n$. $\quad \varphi(C \setminus \{P\}) \cap U \neq \emptyset$.
Set $f_{ij} = \frac{x_i}{x_j} \circ \varphi \in k(C)$. (Def. on $\varphi^{-1}(U) \neq \emptyset$)
$V_P : k(C)^* \longrightarrow \mathbb{Z}$ valuation given by $\mathcal{O}_{C,P}$.
Set $r_i = V_P(f_{i0})$, $0 \le i \le n$.
Choose j s.t. r_j is minimal.

$$\frac{x_i}{x_j} = \frac{x_i/x_0}{x_j/x_0} \Rightarrow f_{ij} = f_{i0}/f_{j0} \Rightarrow V_P(f_{ij}) = r_i - r_j \ge 0.$$
$$\Rightarrow f_{ij} \in \mathcal{O}_{C,P} \ \forall i.$$

<u>Note</u>: If $Q \in \varphi^{-1}(U)$ then $\varphi(Q) = (f_{0j}(Q) : f_{1j}(Q) : \cdots : f_{jj}(Q) = 1 : \cdots : f_{nj}(Q))$
Extend φ by defining $\varphi(P)$ by this expression.
Then φ morphism on $\varphi^{-1}(U) \cup \{P\}$ (which is open!)
So $\varphi : C \longrightarrow \mathbb{P}^n$ morphism.
<u>Unique</u>: $\mathbb{P}^n$ separated.
□

<u>Lemma</u> $R \subseteq K$ local ring, $k \subseteq R$, R not a field. Then R is contained in some DVR of K/k.

<u>Proof</u> Set $B = \overline{R} \subseteq K$. Lying over $\Rightarrow \exists$ max ideal $M \subseteq B$ s.t. $m_R = R \cap M$.
<u>Claim</u>: B_M DVR of K/k.

Let $0 \neq f \in \mathfrak{m}_R$.

$S = \overline{k[f]} \subseteq K$ Dedekind domain and $S \subseteq B$.

$\tilde{M} = M \cap S$ max ideal of S (since $\dim(S) = 1$ and $f \in \tilde{M}$)

$\Rightarrow S_{\tilde{M}}$ DVR of K/k.

$S_{\tilde{M}} \subseteq B_M \subsetneq K \Rightarrow S_{\tilde{M}} = B_M$.

$\square$

Thm C_K is projective

Proof Let $f \in K \setminus k$, $U = \mathrm{Specm}(\overline{k[f]})$, $V = \mathrm{Specm}(\overline{k[f^{-1}]})$.

$C_K = U \cup V$ open affine cover.

$U \subseteq \mathbb{A}^N$ closed. $\bar{U} \subseteq \mathbb{P}^N$ projective closure.

Prop $\Rightarrow$ inclusion $U \longrightarrow \bar{U}$ extends to $\varphi_1 : C_K \longrightarrow \bar{U}$.

Similarly: $\bar{V}$ = proj. closure of V, $V \to \bar{V}$ extends to $\varphi_2 : C_K \to \bar{V}$.

Def $\varphi : C_K \longrightarrow \bar{U} \times \bar{V}$, $\varphi(P) = (\varphi_1(P), \varphi_2(P))$

Set $Y = \overline{\varphi(C_K)} \subseteq \bar{U} \times \bar{V}$.

Then Y irred. projective curve.

Claim: $\varphi : C_K \xrightarrow{\cong} Y$ isomorphism.

Note: $\varphi(U)$ closed in $U \times \bar{V}$:

Let $\psi = \varphi_2 \times id : U \times \bar{V} \longrightarrow \bar{V} \times \bar{V}$.

$\varphi(U) = \{(u,v) \in U \times \bar{V} : \varphi_2(u) = v\} = \psi^{-1}(\Delta_{\bar{V}})$ closed.

$\Rightarrow \varphi(U) = Y \cap (U \times \bar{V})$

$\Rightarrow \varphi : U \xrightarrow{\cong} Y \cap (U \times \bar{V})$ bijective, isomorphism.

inverse: $\pi_U : Y \cap (U \times \bar{V}) \longrightarrow U$.

Similarly: $\varphi : V \longrightarrow Y \cap (\bar{U} \times V)$ isomorphism.

Note: $k(Y) = k(C_K) = K$ and $\forall P \in C_K$ we have $\mathcal{O}_{Y,\varphi(P)} = R_P \subseteq K$.

$\Rightarrow \varphi$ is injective.

Surjective: Let $y \in Y$. Then $\mathcal{O}_{Y,y} \subseteq R_P$ for some $P \in C_K$.

But then $\mathcal{O}_{Y,y} \subseteq R_P = \mathcal{O}_{Y,\varphi(P)} \Rightarrow y = \varphi(P)$.

$\square$

K function field of dim 1 over k.

Thm $C_K = \{\text{DVRs } R_p \text{ of } K/k\}$ non-sing curve with $k(C_K) = K$.

For $p \in C_K$, $f \in R_p$: $f(p) =$ image of f under $R_p \to R_p/m_p = k$.

For $\emptyset \neq U \subseteq C_k$ open, $k[U] = \bigcap_{p \in U} R_p \subseteq K$.

Prop C curve, $P \in C$ non-sing point, Y projective.
Any $\varphi : C \setminus \{P\} \to Y$ can be ext. to $\varphi : C \to Y$.

Lemma $R \subseteq K$ local ring, $k \subseteq R$, R not a field. Then $R \subseteq$ some DVR of K/k

Thm C_K is projective.

Proof Let $f \in K \setminus k$, $U = \operatorname{Spec}(\overline{k[f]})$, $V = \operatorname{Spec}(\overline{k[f^{-1}]})$.
$C_k = U \cup V$ open affine cover.
$U \subseteq \mathbb{A}^N$ closed. $\bar{U} \subseteq \mathbb{P}^N$ projective closure.
Prop $\Rightarrow$ inclusion $U \to \bar{U}$ extends to $\varphi_1 : C_K \to \bar{U}$.
Similarly: $\bar{V} =$ proj. closure of V, $V \to \bar{V}$ ext. to $\varphi_2 : C_K \to \bar{V}$.
Def $\varphi : C_k \to \bar{U} \times \bar{V}$, $\varphi(P) = (\varphi_1(P), \varphi_2(P))$.
Set $Y = \overline{\varphi(C_K)} \subseteq \bar{U} \times \bar{V}$. Then Y proj. variety.
Claim $\varphi : C_k \xrightarrow{\cong} Y$ isomorphism.
Note: $\varphi(U)$ closed in $U \times \bar{V}$:
Let $\psi = \varphi_2 \times id : U \times \bar{V} \to \bar{V} \times \bar{V}$.
$\varphi(U) = \{(u,v) \in U \times \bar{V} \mid \varphi_2(u) = v\} = \psi^{-1}(\Delta_{\bar{V}})$ closed.
$\Rightarrow \varphi(U) = Y \cap (U \times \bar{V})$
$\Rightarrow \varphi : U \xrightarrow{\cong} Y \cap (U \times \bar{V})$ bijective, iso. inverse: $\pi_U : Y \cap (U \times \bar{V}) \to U$.
Similarly: $\varphi : V \xrightarrow{\cong} Y \cap (\bar{U} \times V)$ isomorphism.
Note: $k(Y) = k(C_K) = K$ and $\forall P \in C_K$ we have $\mathcal{O}_{Y,\varphi(P)} = R_p \subseteq K$.
$\Rightarrow \varphi$ is injective.
Surjective: Let $y \in Y$. Then $\mathcal{O}_{Y,y} \subseteq R_p$ for some $P \in C_K$ (Lemma).
But then $\mathcal{O}_{Y,y} \subseteq R_p = \mathcal{O}_{Y,\varphi(P)} \Rightarrow y = \varphi(P)$.
$\square$

Cor 1 Any curve is birat. to non-sing. proj. curve.

Pf X curve $\Rightarrow X \approx C_K$, $K = k(X)$. $\square$

Cor 2 X any non-sing. curve. Then $X \cong$ open $\subseteq C_K$, $K = k(X)$.

Proof Def $\varphi : X \to C_K$, $x \mapsto P$ where $\mathcal{O}_{X,x} = R_P \subseteq K$.

φ clearly injective.

Claim: $\varphi(X) \subseteq C_K$ open.

Take $U \subseteq X$ open affine, $k[U]$ gen. by $f_1, .., f_n$

$$\varphi(U) = \{P \in C_K : k[U] \subseteq R_P\} = \{P \in C_K : f_1, .., f_n \in R_P\} = \bigcap_{i=1}^{n} \operatorname{Spec}\left(\overline{k[f_i]}\right)$$ open.

$\Rightarrow \varphi(X)$ is open.

$\therefore$ $\varphi : X \longrightarrow \varphi(X)$ homeomorphism.

Isomorphism: $U \subseteq X$ open $\Rightarrow$ $k[U] = \bigcap_{x \in U} \mathcal{O}_{X,x} = \bigcap_{P \in \varphi(U)} R_P = k[\varphi(U)]$.

□

Exercise Two non-sing proj. curves are iso $\Leftrightarrow$ same fcn. field.

$\therefore$ One non-sing. proj. curve in each rat. equiv. class of curves.

Degree of proj. variety in $\mathbb{P}^n$

Sometimes useful to compute # points in intersection of "general" subvars $\subseteq \mathbb{P}$

Important concept is <u>degree</u>.

<u>Classical def</u> $X \subseteq \mathbb{P}^n$ closed. $\deg(X) = \#(X \cap V)$

where $V \subseteq \mathbb{P}^n$ generic linear subspace s.t. $\dim(X) + \dim(V) = n$.

E.g. $f \in S = k[x_0, \dots, x_n]$ square-free form $\Rightarrow \#(V_+(f) \cap \text{generic line}) = \deg(f)$.

<u>Warning</u>: $V_+(yz - x^2)$, $V_+(x) \subseteq \mathbb{P}^2$ are isomorphic but have different degs.

<u>Example</u> $f \in S$ square-free deg. d. $X = V_+(f) \subseteq \mathbb{P}^n$

$I(X) = (f)$. $R = S/(f)$ proj. coord. ring of X.

Note: $\dim_k(S_m) = \binom{n+m}{n} = \frac{1}{n!}(m+n)(m+n-1)\cdots(m+1)$

$= \text{polynomial in } m \text{ of deg. } n$, lead coef. $= \frac{1}{n!}$

$0 \to S \xrightarrow{\cdot f} S \to R \to 0 \Rightarrow 0 \to S_{m-d} \to S_m \to R_m \Rightarrow 0$

$\dim(R_m) = \dim(S_m) - \dim(S_{m-d}) = \binom{n+m}{n} - \binom{n+m-d}{n}$

$= \frac{1}{n!}(m+n)\cdots(m+1) - \frac{1}{n!}(m+n-d)\cdots(m+1-d)$

$= \text{poly of deg. } n-1.$

lead coef. $= \frac{1}{n!}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n}(i-d)\right) = \frac{nd}{n!} = \frac{d}{(n-1)!}$.

$\therefore$ Can recover $\deg(f)$ from $m \mapsto \dim_k(R_m)$. <u>Hilbert fcn.</u>

<u>Recall</u>: A graded S-module is a module M with decomp. $M = \bigoplus_{d \in \mathbb{Z}} M_d$ as abelian group, s.t. $S_m \cdot M_d \subseteq M_{m+d}$.

<u>Def</u> $\mathrm{Ann}(M) = \{f \in S \mid f \cdot M = 0\} \subseteq S$ homogeneous ideal.

$\mathrm{Supp}(M) = V_+(\mathrm{Ann}(M)) \subseteq \mathbb{P}^n$.

<u>Reason for name</u>: $x \in \mathbb{P}^n$ point, $P = I(\{x\}) \subseteq S$ prime ideal.

$M_P \neq 0 \iff P \supseteq \mathrm{Ann}(M) \iff x \in \mathrm{Supp}(M)$.

<u>Example</u> $X \subseteq \mathbb{P}^n$ closed. $\mathrm{Ann}(S/I(X)) = I(X) \Rightarrow \mathrm{supp}(S/I(X)) = V_+(I(X)) = X$.

<u>Note</u>: M_d is a k-vectorspace for each d, since $S_0 \cdot M_d \subseteq M_d$.

If M f.g. graded S-module then $\dim_k(M_d) < \infty \quad \forall d$.

(M gen. by $m_1, \dots, m_t$, $m_i \in M_{d_i} \Rightarrow \bigoplus_{i=1}^{t} S_{d-d_i} \xrightarrow{(m_i)} M_d$.

<u>Def</u> $H_M(d) = \dim_k(M_d)$ <u>Hilbert function</u> for M.

<u>Thm</u> M f.g. Graded S-module. Then $\exists !\ P_M(z) \in \mathbb{Q}[z]$ s.t.
$\dim_k(M_d) = P_M(d)$ $\forall\ d \gg 0$. Furthermore: $\deg(P_M) = \dim \operatorname{Supp}(M)$.

<u>Def</u> $P(z) \in \mathbb{Q}[z]$ is a <u>numerical polynomial</u> if $P(d) \in \mathbb{Z}$ $\forall\ d \in \mathbb{Z}$, $d \gg 0$.

<u>Example</u> $\binom{z}{m} = \frac{1}{m!} z(z-1)\cdots(z-m+1)$.

<u>Note</u>: $\{ \binom{z}{m} : m \in \mathbb{N} \}$ basis for $\mathbb{Q}$-vector space $\mathbb{Q}[z]$.

<u>Lemma</u> Let $P(z) = c_0 + c_1\binom{z}{1} + \cdots + c_r\binom{z}{r} \in \mathbb{Q}[z]$, $c_i \in \mathbb{Q}$.

TFAE:
(1) $P(z)$ numerical polynomial
(2) $P(d) \in \mathbb{Z}$ $\forall\ d \in \mathbb{Z}$
(3) $c_i \in \mathbb{Z}$ $\forall\ i$

<u>Proof</u> (3) $\Rightarrow$ (2) $\Rightarrow$ (1) : clear.

Note: $\binom{z+1}{m} - \binom{z}{m} = \binom{z}{m-1}$.

$\Rightarrow P(z+1) - P(z) = c_1 + c_2\binom{z}{1} + \cdots + c_r\binom{z}{r-1}$

(1) $\Rightarrow$ (3):
$P(z)$ num. poly $\Rightarrow$ $P(z+1) - P(z)$ num poly.
Induction $\Rightarrow$ $c_1, c_2, \ldots, c_r \in \mathbb{Z}$.
Must have $c_0 \in \mathbb{Z}$.
$\square$

~~Exmp~~

<u>Def</u> M graded S-module, $M = \bigoplus_{d\in\mathbb{Z}} M_d$. Let $m \in \mathbb{Z}$.
The <u>twisted module</u> $M(m)$ ~~is~~ is the same S-module, with grading $M(m)_d = M_{d+m}$.

<u>Prop</u> M f.g. graded S-module,
There exists a filtration $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$ by graded submodules, s.t. $M^i/M^{i-1} \cong (S/P_i)(\ell_i)$ $\forall i$,
$P_i \subseteq S$ hom. prime ideal, $\ell_i \in \mathbb{Z}$.

<u>Proof</u>
Let $M' \subseteq M$ be a max. graded submodule with such a filtration.
(M Noetherian).
Set $M'' = M/M'$.
If $M'' = 0$: done.
If $M'' \neq 0$:
Choose $x \in M''$ such that $\mathrm{Ann}(x) \subseteq S$ is maximal.
<u>Check</u> $P = \mathrm{Ann}(x) \subseteq S$ hom. prime ideal.

Now $R\cdot x \subseteq M''$ graded submodule.
$R\cdot x \cong S/P(\ell)$, $\ell = -\deg(x)$.

$M \longrightarrow M''$

$N \subseteq M$ inverse img. of $Rx \subseteq M'$.
Then $N \supsetneq M'$. $\lightning$.
$\square$

<u>Affine Dim. Thm</u> $X, Y \subseteq \mathbb{A}^n$ closed, irred. $Z \subseteq X \cap Y$ component.
Then $\dim(Z) \geq \dim(X) + \dim(Y) - n$.

<u>Proof</u> $X \cap Y = (X \times Y) \cap \Delta_{\mathbb{A}^n} = (X \times Y) \cap V(x_1 - y_1, \dots, x_n - y_n) \subseteq \mathbb{A}^n \times \mathbb{A}^n$. □

<u>WARNING</u>: This does not prevent $X \cap Y = \emptyset$!

<u>Proj. Dim. Thm</u> $X, Y \subseteq \mathbb{P}^n$ closed, irred. $Z \subseteq X \cap Y$ component.
Then $\dim(Z) \geq \dim(X) + \dim(Y) - n$. If $\dim(X) + \dim(Y) - n \geq 0$ then $X \cap Y \neq \emptyset$.

<u>Proof</u> First statement: restrict to $D_+(x_i)$ s.t. $D_+(x_i) \cap Z \neq \emptyset$.
Set $s = \dim(X)$, $t = \dim(Y)$. Assume $s + t - n \geq 0$.
$C(X) := \overline{\pi^{-1}(X)} \subseteq \mathbb{A}^{n+1}$. $\dim C(X) = s+1$, $\dim C(Y) = t+1$.
Every comp. of $C(X) \cap C(Y)$ has dim $\geq (s+1) + (t+1) - (n+1) \geq 1$.
And $0 \in C(X) \cap C(Y) \neq \emptyset$. $\Rightarrow$ $X \cap Y \neq \emptyset$. □

$S = \bigoplus_{m \geq 0} S_m$ graded ring, $M = \bigoplus_{d \in \mathbb{Z}} M_d$ graded S-module.

<u>Def</u> Let $\ell \in \mathbb{Z}$. $M(\ell)$ is the <u>twisted module</u> def. by $M(\ell)_d = M_{\ell+d}$.
I.e. same S-module, different grading.

An S-homomorphism $\varphi: M \longrightarrow N$ of graded S-modules is <u>homogeneous</u> if $\varphi(M_d) \subseteq N_d$.

A submodule $N \subseteq M$ is <u>homogeneous</u> if $N = \bigoplus (N \cap M_d)$.
This implies that N and $M/N = \bigoplus \left(M_d / N \cap M_d \right)$ are ~~homogeneous~~ graded, <u>and</u>
$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ short exact with hom. maps.

<u>Note</u> Let $m \in M$ be homogeneous of deg ℓ, i.e. $m \in M_\ell$.
Then $I = \mathrm{Ann}(m) = \{ f \in S : f \cdot m = 0 \} \subseteq S$ homogeneous ideal.
Set $N = S \cdot m \subseteq M$.
$0 \rightarrow I(-\ell) \longrightarrow S(-\ell) \xrightarrow{\cdot m} N \longrightarrow 0 \quad \Rightarrow \quad (S/I)(-\ell) \cong N \subseteq M$.

<u>Exercise</u> M f.g. module over Noetherian ring.
Then M is a <u>Noeth. module</u>, i.e. every submodule is f.g.

<u>Lemma</u> M f.g. graded module over Noeth. graded ring S.
Then $\exists$ filtration $0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$ s.t. $M^i \subseteq M$ homogeneous,
and $M^i / M^{i-1} \cong S/p_i(\ell_i)$ where $p_i \subseteq S$ hom. prime and $\ell_i \in \mathbb{Z}$.

<u>Proof</u> Let $N \subseteq M$ be a max. submodule s.t. Lemma true for N.

Claim: $N = M$.

Else $M'' = M/N \neq 0$.

Choose $0 \neq m \in M''$ homogeneous s.t. $\mathrm{Ann}(m) \subseteq S$ largest possible.

Claim: $P = \mathrm{Ann}(m)$ prime ideal.

Let $f, g \in S \setminus P$ be homogeneous. Enough to show $fg \notin P$.

Note that $g \cdot m \neq 0$ and $\mathrm{Ann}(gm) \supseteq \mathrm{Ann}(m) \Rightarrow \mathrm{Ann}(gm) = \mathrm{Ann}(m)$.

$f \notin \mathrm{Ann}(gm) \Rightarrow fg \notin \mathrm{Ann}(m)$.

$\therefore \quad S \cdot m \cong (S/P)(-\ell)$, $\quad \ell = \deg(m)$.

$M \twoheadrightarrow M/N = M'' \supseteq S \cdot m$.

Let $\tilde{N} \subseteq M$ be inverse image of $S \cdot m$. Then $\tilde{N} \supsetneq N$ and Lemma true for $\tilde{N}$. $\lightning$

$\square$

<u>Def</u> $f : \mathbb{Z} \to \mathbb{Z}$ is <u>eventually polynomial</u> if $\exists\, P(z) \in \mathbb{Q}[z]$ s.t. $f(n) = P(n)$ $\forall\, n \gg 0$.

Set $\Delta f(n) = f(n+1) - f(n)$.

<u>Lemma</u> f event. poly of deg. $r \iff \Delta f$ event. poly of deg $r-1$.

<u>Proof</u> $\Rightarrow$: trivial.

$\Leftarrow$: Assume $\Delta f(n) = Q(n)$ $\forall\, n \gg 0$, where $Q(z) = c_1 + c_2\binom{z}{1} + \cdots + c_r\binom{z}{r-1} \in \mathbb{Q}[z]$

Set $P(z) = c_1\binom{z}{1} + c_2\binom{z}{2} + \cdots + c_r\binom{z}{r}$.

$\Delta P = Q \Rightarrow \Delta(f-P)(n) = 0 \quad \forall\, n \gg 0 \quad \Rightarrow f(n) - P(n) = c_0$ const. $\forall n \gg 0$.

$\therefore$ $f(n)$ eventually $= P(n) + c_0$.

$\square$

Set $S = k[x_0, .., x_n]$, $\quad M$ f.g. graded S-module.

Hilbert fcn: $\quad H_M(d) = \dim_k(M_d)$.

$\mathrm{Supp}(M) = V_+(\mathrm{Ann}(M)) \subseteq \mathbb{P}^n$.

<u>Note</u>: If $0 \to M' \to M \to M'' \to 0$ short exact of graded S-modules, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

$\supseteq$: Follows since $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(M') \cap \mathrm{Ann}(M'')$.

$\subseteq$: Let $x \in \mathbb{P}^n$. If $x \notin \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$ then $\exists\, f \in \mathrm{Ann}(M')$, $g \in \mathrm{Ann}(M'')$ s.t. $f(x) \neq 0$, $g(x) \neq 0$.

But then $fg \in \mathrm{Ann}(M)$ and $fg(x) \neq 0$.

<u>Thm</u> M f.g. graded module over $S = k[x_0, .., x_n]$. Then $H_M(d) = \dim_k(M_d)$ is event. equal to poly $P_M(d)$ with $\deg(P_M) = \dim(\mathrm{Supp}(M))$.

<u>Proof</u> : $\exists$ filtration $0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$, s.t. $M^i/M^{i-1} = (S/P_i)(\ell_i)$ ③

Note: $H_M(d) = \sum_{i=1}^{r} H_{M^i/M^{i-1}}(d) = \sum_{i=1}^{r} H_{S/P_i}(d+\ell_i)$

and $\mathrm{Supp}(M) = \bigcup_{i=1}^{r} \mathrm{supp}(M^i/M^{i-1}) = \bigcup_{i=1}^{r} V_+(P_i)$

WLOG: $M = S/P$, P hom. prime.

<u>Induction on $\dim(V_+(P))$</u> :

If $P = (x_0, x_1, \dots, x_n) \subseteq S$ then $V_+(P) = \emptyset$ and $H_{S/P}(d) = 0 \quad \forall d \geq 1$.

Thm true if we set $\dim(\emptyset) = \deg(0) = -1$.

Otherwise some $x_i \notin P$.

Set $I = (x_i) + P \subseteq S$. Then $V_+(I) \subsetneq V_+(P)$.

Proj. dim. Thm $\Rightarrow$ $\dim V_+(I) = \dim V_+(P) - 1$.

Induction $\Rightarrow$ $H_{S/I}(d)$ event. polynomial, and $\deg(P_{S/I}) = \dim V_+(P) - 1$.

$0 \to (S/P)(-1) \xrightarrow{\cdot x_i} S/P \longrightarrow S/I \to 0 \quad \Rightarrow \Delta H_{S/P}(d-1) = H_{S/P}(d) - H_{S/P}(d-1) = H_{S/I}(d)$

$\Rightarrow$ $H_{S/P}(d)$ event. poly of deg. $= \dim V_+(P)$.

□

<u>Note</u> $P_M(z) = c_0 + c_1\binom{z}{1} + \cdots + c_r\binom{z}{r} \in \mathbb{Q}[z]$ numeric polynomial $\Rightarrow c_i \in \mathbb{Z}$.

$r = \dim(\mathrm{Supp}(M))$. $\quad r! \cdot \text{lead. coef}(P_M) \in \mathbb{Z}$!

<u>Def</u> $X \subseteq \mathbb{P}^n$ closed, $\dim(X) = r$. $P_X(z) = P_{S/I(X)}(z)$ <u>Hilbert poly</u> for X.

$\deg(X) := r! \cdot$ lead. coef of $P_X(z)$.

<u>Examples</u> 1) $\deg V_+(f) = \deg(f)$.

2) $V \subseteq \mathbb{P}^n$ linear subspace of dim. r. WLOG $V = V_+(x_{r+1}, \dots, x_n)$.

$S/I(V) \cong k[x_0, \dots, x_r]$. $P_V(z) = \binom{z+r}{r}$. $\deg(V) = 1$.

<u>Prop</u> $X_1, X_2 \subseteq \mathbb{P}^n$ closed, $\dim(X_1) = \dim(X_2) = r$, no common components.

Then $\deg(X_1) + \deg(X_2) = \deg(X_1 \cup X_2)$.

<u>Proof</u> $I_1 = I(X_1)$, $I_2 = I(X_2)$. $I(X_1 \cup X_2) = I_1 \cap I_2$.

$$0 \to S/I_1 \cap I_2 \longrightarrow S/I_1 \oplus S/I_2 \longrightarrow S/I_1 + I_2 \to 0$$

$$f \mapsto (f, f) \qquad (f, g) \mapsto f - g$$

$\Rightarrow P_{X_1}(d) + P_{X_2}(d) = P_{X_1 \cup X_2}(d) + P_{S/I_1+I_2}(d)$.

$\deg(P_{S/I_1+I_2}) = \dim V_+(I_1 + I_2) = \dim(X_1 \cap X_2) < r$

$\Rightarrow$ $\text{l.c.}(P_{X_1}) + \text{l.c.}(P_{X_2}) = \text{l.c.}(P_{X_1 \cup X_2}) \Rightarrow \deg(X_1) + \deg(X_2) = \deg(X_1 \cup X_2)$.

□

Cor If $X \subseteq \mathbb{P}^n$ has dim. zero, then $\deg(X) = \# X$.

Length R ring, M R-module.

M is *simple* if M has no non-trivial submodules, and $M \neq 0$.

Equivalently: $M \cong R/\mathfrak{p}$, $\mathfrak{p} \subseteq R$ max. ideal.

A *decomposition series* for M is a filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ s.t. M_i/M_{i-1} simple $\forall i$.

Def $\exists$ decomp series $\Rightarrow$ M is *Artinian* and $\mathrm{length}(M) = r$.

Assume R Noetherian, M f.g. R-module.

Lemma $\Rightarrow$ $\exists \; 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ (⊛) s.t. $M_i/M_{i-1} = R/P_i$, P_i prime.

Note: $P_i \supseteq \mathrm{Ann}(M)$ $\forall i$.

Lemma If $P \subseteq R$ min. prime over $\mathrm{Ann}(M)$, then M_P Artinian R_P-module, and $\mathrm{length}_{R_P}(M_P) = \#\{ i : P_i = P \text{ in } (⊛) \}$.

Proof

$$0 = (M_0)_P \subseteq (M_1)_P \subseteq \cdots \subseteq (M_r)_P = M_P$$

$$(M_i)_P/(M_{i-1})_P = (M_i/M_{i-1})_P = (R/P_i)_P = \begin{cases} R_P/P_P & \text{if } P = P_i \\ 0 & \text{if } P \neq P_i \ (\Rightarrow P_i \not\subseteq P) \end{cases}$$

□

$X \subseteq \mathbb{P}^n$ closed of pure dim. r.

$Y = V_+(f) \subseteq \mathbb{P}^n$ hypersurface s.t. no comp. of X is inside Y.

Assume $f \in S$ square free.

Let $Z \subseteq X \cap Y$ be a component. Then $\dim(Z) = r-1$.

Set $M = S/(I(X) + (f))$.

$\mathrm{Supp}(M) = X \cap Y \Rightarrow P = I(Z)$ is min. over $\mathrm{Ann}(M)$.

Def $I(X \cdot Y; Z) = \mathrm{length}_{S_P}(M_P)$.

Thm $$\deg(X) \cdot \deg(Y) = \sum_{Z \subseteq X \cap Y \text{ comp}} I(X \cdot Y; Z) \cdot \deg(Z).$$

Length R ring, M R-module.
M is *simple* if M has no non-trivial submodules and $M \neq 0$.
Equivalent: $M \cong R/p$, $p \subseteq R$ max ideal.

A *decomposition series* for M is a filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$
s.t. M_i/M_{i-1} simple $\forall i$.

Def $\exists$ decomp. series $\Rightarrow$ M is Artinian and $\mathrm{length}(M) = r$.

Assume R Noetherian, M f.g. R-module.

Lemma $\Rightarrow$ $\exists$ $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ ⊛ s.t. $M_i/M_{i-1} \cong R/P_i$, P_i prime.

Note: $\mathrm{Ann}(M) \subseteq P_i$ $\forall i$.

Lemma If $P \subseteq R$ min prime over $\mathrm{Ann}(M)$, then M_P Artinian R_P-module, and
$$\mathrm{length}_{R_P}(M_P) = \#\{i : P_i = P \text{ in } ⊛\}$$

Proof
$$0 = (M_0)_P \subseteq (M_1)_P \subseteq \cdots \subseteq (M_r)_P = M_P$$
$$(M_i)_P/(M_{i-1})_P = (M_i/M_{i-1})_P = (R/P_i)_P = \begin{cases} R_P/PR_P & \text{if } P = P_i \\ 0 & \text{if } P \neq P_i \quad (\Rightarrow P_i \not\subseteq P) \end{cases}$$
□

$X \subseteq \mathbb{P}^n$ closed. $S = k[x_0, \ldots, x_n]$.

$\dim_k (S/I(X))_d = P_X(d)$ for $d >> 0$, $P_X(z)$ Hilbert poly

$\deg(P_X) = \dim(X)$, $\deg(X) := (\dim(X)!) \cdot \text{lead coef.}(P_X)$.

Assume $X \subseteq \mathbb{P}^n$ has pure dim. r.

$Y = V_+(f) \subseteq \mathbb{P}^n$ hypersurface s.t. no comp. of X is $\subseteq Y$.

Assume $f \in S$ square free.

Let $Z \subseteq X \cap Y$ be a component. Then $\dim(Z) = r-1$.

Set $M = S/I(X) + (f)$.

$\mathrm{Supp}(M) = X \cap Y \Rightarrow P = I(Z)$ min. prime over $\mathrm{Ann}(M)$.

Def $I(X \cdot Y; Z) = \mathrm{length}_{S_P}(M_P)$

Thm
$$\deg(X) \cdot \deg(Y) = \sum_{\substack{Z \subseteq X \cap Y \\ \text{comp}}} I(X \cdot Y; Z) \cdot \deg(Z).$$

Proof
Set $d = \deg(X)$, $e = \deg(Y)$.

$0 \to S/I(X) \xrightarrow{\cdot f} S/I(X) \to M \to 0$ is exact (If $h \cdot f \in I(X)$ then $h = 0$ on each comp. of X.)

$0 \to (S/I(X))_{\ell - e} \to (S/I(X))_\ell \to M_\ell \to 0$

$\Rightarrow \; P_M(\ell) = P_X(\ell) - P_X(\ell - e)$

$\Rightarrow \; \text{l.c.}(P_M) = r \cdot e \cdot \text{l.c.}(P_X)$ $\quad \left[z^r - (z-e)^r = r e z^{r-1} + \cdots \right]$

$= \frac{red}{r!} = \frac{ed}{(r-1)!}$

Take filtration: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$, $M_i \overset{\subseteq M}{}$ homogeneous, $M_i/M_{i-1} \cong S/P_i$, $P_i \subseteq S$ prime.

$$P_M(\ell) = \sum_{i=1}^{t} P_{M_i/M_{i-1}}(\ell) = \sum_{\substack{Q \subseteq R \\ \text{prime}}} \#\{i : P_i = Q\} \cdot P_{S/Q}(\ell)$$

$$= \sum_{\substack{Z \subseteq X \cap Y \\ \text{comp.}}} \underbrace{\#\{i : P_i = I(Z)\}}_{I(X \cdot Y; Z)} \cdot P_Z(\ell) \quad + \text{ lower terms.}$$

$\therefore \; \text{l.c.}(P_M) = \frac{1}{(r-1)!} \sum_{Z \subseteq X \cap Y} I(X \cdot Y; Z) \cdot \deg(Z).$

□

__Cor__ (Bézout's Thm) $X, Y \subseteq \mathbb{P}^2$ curves of degree d, e s.t. $X \cap Y = \{P_1, \dots, P_m\}$ finite.

Then $\sum_{i=1}^{m} I(X \cdot Y; P_i) = de.$

__Exercise__ $P \in X \cap Y \subseteq \mathbb{P}^2$.

$I(X \cdot Y; P) = 1 \iff X$ and Y both non-sing at P, and tangent directions are different.

__Bezout's Thm for $\mathbb{P}^n$:__

__Idea:__ If $X \subseteq \mathbb{P}^n$ irred. and $Y \subseteq \mathbb{P}^n$ hypersurface s.t. $X \not\subseteq Y$, set

$$[X] \cdot [Y] = \sum_{\substack{Z \subseteq X \cap Y \\ \text{comp}}} I(X \cdot Y; Z) \cdot [Z].$$

Suppose $Y_1, Y_2, \dots, Y_n \subseteq \mathbb{P}^n$ hypersurfaces s.t. $\#(Y_1 \cap \cdots \cap Y_n) < \infty$.

Compute formally: $(\cdots(([\mathbb{P}^n] \cdot [Y_1]) \cdot [Y_2]) \cdots) \cdot [Y_n] = \sum_{i=1}^{N} c_i \cdot [P_i]$

where $Y_1 \cap \cdots \cap Y_n = \{P_1, \dots, P_N\}$.

Then $\sum c_i = \prod_{j=1}^{n} \deg(Y_j)$.

In fact: $(\cdots(\mathbb{P}^n \cdot [Y_1]) \cdots) \cdot [Y_m] = \sum_{i=1}^{N_m} c_i^{(m)} [Z_i]$

$Y_1 \cap \cdots \cap Y_m = Z_1 \cup \cdots \cup Z_{N_m}$ comps, $\quad \prod_{j=1}^{m} \deg(Y_j) = \sum c_i^{(m)} \deg(Z_i).$

__Fact__ c_i (and $c_i^{(m)}$) well defined, i.e. indep. of order of mult.

__Cor__ $\#(Y_1 \cap \cdots \cap Y_n) < \infty \Rightarrow \#(Y_1 \cap \cdots \cap Y_n) \leq \prod_{j=1}^{n} \deg(Y_j).$

<u>Useful fact</u> $X_1, \dots, X_m \subseteq \mathbb{P}^n$ irred. closed sets, $d \in \mathbb{N}$.
Then $\exists$ irred. hypersurface Y of degree d s.t. $X_i \nsubseteq Y$ $\forall i$.

$v_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ Veronese, $N = \binom{n+d}{n} - 1$.

$Y = \mathbb{P}^n \cap H$, $H \subseteq \mathbb{P}^N$ hyperplane.

<u>Intrinsic</u>: $\{\text{hypersurfaces of deg } d \text{ in } \mathbb{P}^n\} \leftrightarrow \mathbb{P}(S_d) \setminus \{\text{non-square free forms}\}$

$$V_+(f) \leftrightarrow [f].$$

$$\{\text{irred. hypersurfaces}\} \leftrightarrow U = \mathbb{P}(S_d) \setminus \bigcup_{p+q=d} \varphi_{p,q}(\mathbb{P}(S_p) \times \mathbb{P}(S_q))$$

where $\varphi_{p,q} : \mathbb{P}(S_p) \times \mathbb{P}(S_q) \longrightarrow \mathbb{P}(S_d)$, $\varphi_{p,q}([f],[g]) = [fg]$.

$\binom{n+p}{n} + \binom{n+q}{n} \leq \binom{n+p+q}{n} \Rightarrow U \subseteq \mathbb{P}(S_d)$ dense open.

$X \subseteq \mathbb{P}^n$ closed. $X \subseteq V_+(f) \Leftrightarrow f \in I(X)_d$

$\{\text{hypersurfaces} \not\supseteq X\} \leftrightarrow W_X = \mathbb{P}(S_d) \setminus \mathbb{P}(I(X)_d)$ dense open.

<u>Conclude</u>: $\{\text{irred. hypersurf. of deg. } d \not\supseteq X_i \ \forall i\} \leftrightarrow U \cap W_{X_1} \cap \dots \cap W_{X_m} \subseteq \mathbb{P}(S_d)$ dense open.

<u>Sheaves</u>

<u>Def</u> X top. space. A <u>presheaf</u> $\mathcal{F}$ of abelian groups on X is an assignment $U \mapsto \mathcal{F}(U)$ of an abelian group $\mathcal{F}(U)$ to each open $U \subseteq X$, plus group homomorphisms $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$ s.t.

(1) $\rho_{UU} = \mathrm{id} : \mathcal{F}(U) \to \mathcal{F}(U)$

(2) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ if $W \subseteq V \subseteq U$

<u>Notation</u> Elements of $\mathcal{F}(U)$ are called <u>sections</u> of $\mathcal{F}$ over U.

ρ_{UV} = restriction map.

$s|_V = \rho_{UV}(s)$ if $s \in \mathcal{F}(U)$.

$\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

<u>Def</u> A <u>sheaf</u> is a presheaf $\mathcal{F}$ s.t.

(3) If $s \in \mathcal{F}(U)$ and $U = \bigcup V_i$ open cover then $s = 0 \Leftrightarrow s|_{V_i} = 0$ $\forall i$

(4) If $U = \bigcup V_i$ open cover and we have sections $s_i \in \mathcal{F}(V_i)$ $\forall i$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ $\forall i,j$ then $\exists s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$ $\forall i$.

<u>Note</u> (3) $\Rightarrow$ section s of (4) unique.

<u>Point</u>: Global data determined by local data.

<u>Remark</u>: Also have sheaves of rings, modules, sets, etc.

<u>Examples</u> 1) X swF. Def sheaf of k-algebras $\mathcal{O}_X$ by $\mathcal{O}_X(U) = k[U]$.
$\mathcal{O}_X$ = structure sheaf of X.

2) M manifold. $\mathcal{O}_M(U) = \{C^\infty \text{ fcn } f: U \to \mathbb{R}\}$

$TM \xrightarrow{\pi} M$ tangent bundle.

I.e. TM manifold, π C^∞-map s.t. $\pi^{-1}(x) = T_x M$ $\forall x \in M$.

E.g. If $M \subseteq \mathbb{R}^n$ then $TM = \{(x, \vec{v}) \in M \times \mathbb{R}^n : \vec{v} \text{ tangent direction at } x\}$

$TM \leftrightarrow$ sheaf $\mathcal{T}$ of $\mathcal{O}_M$-modules.

$\mathcal{T}(U) = \{\text{sections } s: U \to TM : \pi(s(x)) = x \ \forall x \in U\}$

$\mathcal{T}(U)$ is a $\mathcal{O}_M(U)$-module: $(f \cdot s)(x) = f(x) \cdot s(x) \in T_x M$.

<u>Def</u> A <u>morphism</u> of (pre)sheaves $\varphi: \mathcal{F} \to \mathcal{G}$ consists of homomorphisms $\varphi_U: \mathcal{F}(U) \to \mathcal{G}(U)$ $\forall U \subseteq X$ open s.t. $s \in \mathcal{F}(U)$, $V \subseteq U \Rightarrow \varphi_U(s)|_V = \varphi_V(s|_V)$.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V) \end{tikzcd}$$

If $\varphi: \mathcal{F} \to \mathcal{G}$ and $\psi: \mathcal{G} \to \mathcal{H}$ morphisms, def $\psi \circ \varphi: \mathcal{F} \to \mathcal{H}$ by $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$.

<u>isomorphism</u> = morphism with inverse morphism.

X top. space.

$\mathcal{F}$ (pre)sheaf on X: $U \mapsto \mathcal{F}(U)$ abelian group, plus restrictions $\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$, $\sigma \mapsto \sigma|_V$.

presheaf: compatibility of restrictions.

sheaf: decency ~~and~~ patching.

Def A morphism of (pre)sheaves $\varphi: \mathcal{F} \to \mathcal{G}$ consists of homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ $\forall\ U \subseteq X$ open, s.t. $V \subseteq U$, $s \in \mathcal{F}(U) \Rightarrow \varphi_V(s|_V) = \varphi_U(s)|_V$.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V) \end{array}$$

If $\varphi: \mathcal{F} \to \mathcal{G}$ and $\psi: \mathcal{G} \to \mathcal{H}$ morphisms, def. $\psi \circ \varphi : \mathcal{F} \to \mathcal{H}$ by $(\psi\circ\varphi)_U = \psi_U \circ \varphi_U$.

An isomorphism = morphism with inverse morphism.

Def $\mathcal{F}$ presheaf on X, $p \in X$ point. $\mathcal{F}_p = \varinjlim_{U \ni x} \mathcal{F}(U)$ stalk of $\mathcal{F}$ at p.

This means:

1) $\mathcal{F}_p$ = abelian group (or whatever)

2) If $p \in U$ we have hom. $\rho_{Up} : \mathcal{F}(U) \to \mathcal{F}_p$.
   If $p \in V \subseteq U$ then $\rho_{Up} = \rho_{Vp} \circ \rho_{UV}$.

3) If G any abelian gp with homomorphisms $\Theta_U : \mathcal{F}(U) \to G$ $\forall\ U \ni p$, s.t. $\Theta_U = \Theta_V \circ \rho_{UV}$ for $p \in V \subseteq U$, then $\exists!$ hom $\Theta_p : \mathcal{F}_p \to G$ s.t. $\Theta_U = \Theta_p \circ \rho_{Up}$ $\forall\ U \subseteq X$ open, $p \in U$.

Example X variety. $\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U)$ local ring of X at p.

Construction $\mathcal{F}_p = \left(\bigoplus_{U \ni p} \mathcal{F}(U)\right) \Big/ \left\langle (0, \dots, 0, \underset{\mathcal{F}(U)}{s}, \dots, \underset{\mathcal{F}(V)}{-s|_V}, 0, \dots, 0) : \begin{array}{l} s \in \mathcal{F}(U), \\ p \in V \subseteq U \end{array} \right\rangle$

Notation
If $s \in \mathcal{F}(U)$, $p \in U$, then $s_p = \rho_{Up}(s) \in \mathcal{F}_p$.

Exercise $\mathcal{F}$ presheaf.

1) All elements of $\mathcal{F}_p$ can be written as s_p for some $s \in \mathcal{F}(U)$, $p \in U$.

2) $s \in \mathcal{F}(U)$, $p \in U$. $s_p = 0 \in \mathcal{F}_p \Leftrightarrow \exists\ p \in V \subseteq U : s|_V = 0 \in \mathcal{F}(V)$.

Exercise $\mathcal{F}$ sheaf, $s \in \mathcal{F}(U)$. $s = 0 \Leftrightarrow s_p = 0\ \forall p \in U$.

<u>Note</u>: A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ gives a homomorphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$.

$\varphi_p(s_p) = \varphi_U(s)_p$ for $s \in \mathcal{F}(U)$, $p \in U$.

<u>Prop</u> $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of <u>sheaves</u>. Then φ isomorphism $\Leftrightarrow$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ iso $\forall p \in X$.

<u>Proof</u>

$\Rightarrow$: clear.

$\Leftarrow$: Must show $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ iso $\forall$ open $U \subseteq X$.

<u>φ_U injective</u>: If $\varphi_U(s) = 0 \in \mathcal{G}(U)$ then $\varphi_p(s_p) = \varphi_U(s)_p = 0 \;\forall p \in U$

$\Rightarrow s_p = 0 \;\forall p \in U \Rightarrow s = 0 \in \mathcal{F}(U)$.

<u>φ_U surjective</u>: Let $t \in \mathcal{G}(U)$.

φ_p surjective $\Rightarrow t_p = \varphi_p(\, s(p)_p \,) \in \mathcal{G}_p \quad \forall p \in U$

for some $s(p) \in \mathcal{F}(V_p)$, $p \in V_p \subseteq U$.

Now $t_p = \varphi_{V_p}(s(p))_p$. Make V_p smaller s.t. $t|_{V_p} = \varphi_{V_p}(s(p))$.

$\varphi(s(p)|_{V_p \cap V_q}) = t|_{V_p \cap V_q} = \varphi(s(q)|_{V_p \cap V_q})$

$\Rightarrow s(p)|_{V_p \cap V_q} = s(q)|_{V_p \cap V_q} \quad \forall p, q \in U$.

$\therefore \exists!\; s \in \mathcal{F}(U)$ s.t. $s|_{V_p} = s(p) \in \mathcal{F}(V_p) \;\forall p$.

$\varphi_U(s)|_{V_p} = \varphi_{V_p}(s|_{V_p}) = \varphi_{V_p}(s(p)) = t|_{V_p} \Rightarrow \varphi_U(s) = t \in \mathcal{G}(U)$.

$\square$

<u>Remark</u> $\mathcal{F}$ sheaf on X, $U \subseteq X$ open. Def. sheaf $\mathcal{F}|_U$ on U by

$\Gamma(V, \mathcal{F}|_U) = \Gamma(V, \mathcal{F})$.

<u>Examples</u> $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Def. sheaves $\mathcal{F}$ and $\mathcal{G}$ on S^1 as follows.

$\mathcal{F}(U) = \{ f : U \to \mathbb{R} \mid f \text{ locally const.}\}$

$\mathcal{G}(U) = \{ f : U \to \mathbb{R} \mid f \; C^\infty, \; \frac{\partial f}{\partial \theta} = 1 \}$

If $U \subsetneq S^1$ then $\mathcal{F}|_U \cong \mathcal{G}|_U$, $f(x,y) \mapsto f(x,y) + \arg(x,y)$.

In particular: $\mathcal{F}_p \cong \mathcal{G}_p \;\forall p \in S^1$.

BUT $\mathcal{F} \not\cong \mathcal{G}$. In fact $\mathcal{F}(S^1) = \mathbb{R}$, $\mathcal{G}(S^1) = \varnothing$.

Problem: $\nexists$ morphism $\mathcal{F} \to \mathcal{G}$.

Sheafification

$\mathcal{F}$ presheaf on X. Define a sheaf $\mathcal{F}^+$ as follows:

If $U \subseteq X$ open, set $\mathcal{F}^+(U) =$ set of fcns. $s: U \longrightarrow \coprod_{p \in U} \mathcal{F}_p$ s.t.

1) $s(p) \in \mathcal{F}_p$
2) $\forall p \in U$ $\exists$ open V, $p \in V \subseteq U$ and $t \in \mathcal{F}(V)$ s.t. $s(q) = t_q$ $\forall q \in V$.

Def morphism $\theta: \mathcal{F} \longrightarrow \mathcal{F}^+$: If $t \in \mathcal{F}(U)$ then $\theta_U(t) = [p \mapsto t_p] \in \mathcal{F}^+(U)$.

Exercise $\theta_p : \mathcal{F}_p \longrightarrow \mathcal{F}^+_p$ isomorphism $\forall p \in X$.

Prop $\mathcal{F}$ presheaf, $\mathcal{G}$ sheaf, $\varphi: \mathcal{F} \to \mathcal{G}$ morphism.
Then $\exists!$ $\varphi^+ : \mathcal{F}^+ \longrightarrow \mathcal{G}$ s.t. $\varphi = \varphi^+ \circ \theta$.

Proof Let $s \in \mathcal{F}^+(U)$, i.e. $s: U \longrightarrow \coprod \mathcal{F}_p$.
For $p \in U$, choose $t(p) \in \mathcal{F}(V_p)$ s.t. $p \in V_p \subseteq U$ and $s(q) = t(p)_q$ $\forall q \in V_p$.
Set $\tau(p) = \varphi_{V_p}(t(p)) \in \mathcal{G}(V_p)$.
If $q \in V_p$ then $\tau(p)_q = \varphi(t(p))_q = \varphi_q(t(p)_q) = \varphi_q(s(q)) \in \mathcal{G}_q$.
$\Rightarrow \tau(p_1) = \tau(p_2)$ on $V_{p_1} \cap V_{p_2}$ $\Rightarrow$ $\{\tau(p)\}$ glue to $\tau \in \mathcal{G}(U)$.
Set $\varphi^+_U(s) = \tau$.
□

Remark $\mathcal{F}^+$ is uniquely determined by the univ. property of Prop. (up to unique iso)

Def $\varphi: \mathcal{F}' \to \mathcal{F}$ morphism of sheaves.
φ is *injective* if φ_U injective $\forall U \subseteq X$ open.
(Can identify $\mathcal{F}'$ with "subsheaf" of $\mathcal{F}$, $\mathcal{F}'(U) \subseteq \mathcal{F}(U)$.)

Exercise φ is injective $\Leftrightarrow$ $\varphi_p : \mathcal{F}'_p \longrightarrow \mathcal{F}_p$ injective $\forall p \in X$.

Consequence If $\varphi: \mathcal{F} \to \mathcal{G}$ morphism of presheaves s.t. $\varphi_U: \mathcal{F}(U) \to \mathcal{G}(U)$ injective $\forall U$ then $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ is injective.
(Proof: Must check $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ injective $\forall p \in X$.
If $\varphi_p(s_p) = 0 \in \mathcal{G}_p$ ~~then~~ for some $s \in \mathcal{F}(U)$, $p \in U$ then
$\varphi_U(s)_p = 0 \Rightarrow \varphi_U(s)|_V = 0 \Rightarrow \varphi_V(s|_V) = 0 \Rightarrow s|_V = 0 \Rightarrow s_p = 0$.)

In particular: If presheaf $\mathcal{F}$ is a subsheaf of a sheaf $\mathcal{G}$, then $\mathcal{F}^+ \subseteq \mathcal{G}$.

Def $\varphi: \mathcal{F} \to \mathcal{G}$ morphism of sheaves of abelian groups.

$\ker(\varphi)$ = the sheaf $U \mapsto \ker(\varphi_U) \subseteq \mathcal{F}(U)$.

$\operatorname{Im}(\varphi)$ = the sheafification of the presheaf $U \mapsto \operatorname{Im}(\varphi_U) \subseteq \mathcal{G}(U)$.

Note: $\ker(\varphi) \subseteq \mathcal{F}$ and $\operatorname{Im}(\varphi) \subseteq \mathcal{G}$ subsheaves.

φ injective $\Leftrightarrow$ $\ker(\varphi) = 0$.

Def φ is surjective if $\operatorname{Im}(\varphi) = \mathcal{G}$.

Warning: φ surjective $\not\Rightarrow$ φ_U surjective!

Exercise $\operatorname{Im}(\varphi)_p = \varphi_p(\mathcal{F}_p) \subseteq \mathcal{G}_p$.

$\therefore$ φ surjective $\Leftrightarrow$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ surjective $\forall p \in X$.

Def $\mathcal{F}' \subseteq \mathcal{F}$ subsheaf. Set $\mathcal{F}/\mathcal{F}' = \left[U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)\right]^+$.

Note: Have surjective morphism $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}'$, and $\ker(\mathcal{F} \to \mathcal{F}/\mathcal{F}') = \mathcal{F}'$.

Example X variety, $Y \subseteq X$ closed subvariety.

For $U \subseteq X$ open, set $I_Y(U) = \{ f \in \mathcal{O}_X(U) \mid f(y) = 0 \ \forall y \in U \cap Y \}$.

Then $I_Y \subseteq \mathcal{O}_X$ subsheaf of ideals.

$\mathcal{O}_X(U)/I_Y(U) = \{$ regular functions $f: U \cap Y \to k$ that can be extended to all of U. $\}$

We have: $\mathcal{O}_X/I_Y = \mathcal{O}_Y$ structure sheaf of Y.

Reason: Every regular fcn on Y can be extended to X locally.

Notation A seq. $\cdots \to \mathcal{F}^i \xrightarrow{\varphi_i} \mathcal{F}^{i+1} \xrightarrow{\varphi_{i+1}} \mathcal{F}^{i+2} \to \cdots$ is a complex

if $\varphi_{i+1} \circ \varphi_i = 0$.

Exact if $\operatorname{Im}(\varphi_i) = \ker(\varphi_{i+1})$.

Equivalent:

$\to \mathcal{F}^i_p \to \mathcal{F}^{i+1}_p \to \cdots$ ~~...~~ is a complex/exact.

$\big(\operatorname{Im}(\varphi_i) = \ker(\varphi_{i+1}) \Leftrightarrow \operatorname{Im}(\varphi_i)_p = \ker(\varphi_{i+1})_p \Leftrightarrow \operatorname{Im}(\varphi_{i\,p}) = \ker((\varphi_{i+1})_p)$.

Example $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$.

Alg. Geo. 11/6/06.

$\mathcal{F}$ presheaf on X, $U \subseteq X$ open.

$$\mathcal{F}^+(U) = \left\{ s : U \longrightarrow \coprod_{p \in U} \mathcal{F}_p \;\middle|\; s(p) \in \mathcal{F}_p,\ \begin{array}{l} \forall p \in U\ \exists \text{ open } V,\ p \in V \subseteq U \text{ and } t \in \mathcal{F}(V) \\ \text{s.t. } s(q) = t_q \in \mathcal{F}_q\ \ \forall\ q \in V. \end{array} \right\}.$$

Def $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves of abelian groups.

$\ker(\varphi) =$ the sheaf $U \longmapsto \ker(\varphi_U) \subseteq \mathcal{F}(U)$.

$\mathrm{Im}(\varphi) =$ the sheafification of the presheaf $U \mapsto \mathrm{Im}(\varphi_U) \subseteq \mathcal{G}(U)$.

Note: $\ker(\varphi) \subseteq \mathcal{F}$ and $\mathrm{Im}(\varphi) \subseteq \mathcal{G}$ are subsheaves.

φ injective $\Leftrightarrow \ker(\varphi) = 0$.

Def φ is surjective if $\mathrm{Im}(\varphi) = \mathcal{G}$.

Warning: φ surjective $\not\Rightarrow$ φ_U surjective!

Exercise φ surjective $\Leftrightarrow$ $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ surj. $\forall p \in X$.

Def $\mathcal{F}' \subseteq \mathcal{F}$ subsheaf. Set $\mathcal{F}/\mathcal{F}' = \left[U \mapsto \mathcal{F}(U)/\mathcal{F}'(U) \right]^+$.

Note: Have surjective morphism $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}'$, and $\ker(\mathcal{F} \to \mathcal{F}/\mathcal{F}') = \mathcal{F}'$.

~~**Example** X variety, $Y \subseteq X$ closed subvariety.~~

~~For $U \subseteq X$ open, set $I_Y(U) = \{ f \in \mathcal{O}_X(U) \mid f(y) = 0\ \forall\ y \in U \cap Y \}$.~~

~~Then $I_Y \subseteq \mathcal{O}_X$ subsheaf of ideals.~~

~~$\mathcal{O}_X(U)/I_Y(U) = \{$ regular functions $f : U \cap Y \longrightarrow k$ that can be extended to all of U. $\}$~~

~~We have: $\mathcal{O}_X/I_Y \not\cong \mathcal{O}_Y$. structure sheaf of Y ($\mathcal{O}_Y(U) := \mathcal{O}(Y \cap U)$.)~~

~~Reason: Every regular fcn. on Y can be ext. to X locally.~~

Notation: Seq. of sheaves $\cdots \longrightarrow \mathcal{F}^i \xrightarrow{\varphi_i} \mathcal{F}^{i+1} \xrightarrow{\varphi_{i+1}} \mathcal{F}^{i+2} \longrightarrow \cdots$

Complex: $\varphi_{i+1} \circ \varphi_i = 0 \quad \forall i$

Exact: $\mathrm{Im}(\varphi_i) = \ker(\varphi_{i+1}) \quad \forall i$.

Equivalent: $\longrightarrow \mathcal{F}^i_p \longrightarrow \mathcal{F}^{i+1}_p \longrightarrow \mathcal{F}^{i+1}_p \longrightarrow \cdots$ is exact/complex $\forall p \in X$.

Example $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact $\Leftrightarrow$ $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F}/\mathcal{F}' \xrightarrow{\cong} \mathcal{F}''$.

<u>Def</u> $f : X \to Y$ cont. map. of top spaces, $\mathcal{F}$ sheaf on X.
The <u>push forward</u> of $\mathcal{F}$ by f is the <u>sheaf</u> $f_*\mathcal{F}$ on Y: $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$

<u>Example</u> X variety, $Y \subseteq X$ closed, $i : Y \to X$ inclusion.
For $U \subseteq X$ open, set $I_Y(U) = \{ f \in \mathcal{O}_X(U) \mid f(y) = 0 \ \forall\, y \in U \cap Y \}$
Then $I_Y \subseteq \mathcal{O}_X$ subsheaf of ideals.
$\mathcal{O}_X(U)/I_Y(U) = \{$ regular fcns. $U \cap Y \to k$ that can be extended to <u>all</u> of $U \}$
We have $\mathcal{O}_X/I_Y \cong i_*\mathcal{O}_Y$, where $i_*\mathcal{O}_Y(U) = \mathcal{O}_Y(U \cap Y)$.
Reason: Every regular fcn. on Y can be extended to X <u>locally</u>.
Note: $0 \to I_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$
Sloppy notation: $0 \to I_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$.

<u>Example</u> $f : X \to Y$ morphism of SWFs.
Gives morphism $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$.
$\mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$ given by $h \mapsto h \circ f = f^*h$.

<u>Exercise</u> Find morphism of varieties $f : X \to Y$ s.t. $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ iso. but f **not** iso.

<u>Inverse image sheaves.</u>
$f : X \to Y$ cont., $\mathcal{G}$ sheaf on Y. Want sheaf on X!
If $U \subseteq X$ open, set $\text{pre-}f^{-1}\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$.
Born with maps $\mathcal{G}(V) \to \text{pre-}f^{-1}\mathcal{G}(U)$, $s \mapsto f^{-1}s$, for $f(U) \subseteq V$.
$\text{pre-}f^{-1}\mathcal{G}(U) = \{ f^{-1}s \mid s \in \mathcal{G}(V),\ V \supseteq f(U) \}$
$f^{-1}s = 0 \iff s|_W = 0 \in \mathcal{G}(W)$ where $f(U) \subseteq W \subseteq V$.
$\text{pre-}f^{-1}\mathcal{G}$ is a <u>presheaf</u> on X.

<u>Def</u> $f^{-1}\mathcal{G} = (\text{pre-}f^{-1}\mathcal{G})^+$. Inverse image of $\mathcal{G}$ by f.
Special case: If $X \subseteq Y$ subset, $i : X \to Y$ inclusion, then $\mathcal{G}|_X = i^*\mathcal{G}$.

<u>Example</u> $X \subseteq Y$ open subset.
Then $\text{pre-}i^{-1}\mathcal{G}(U) = \varinjlim_{V \supseteq U} \mathcal{G}(V) = \mathcal{G}(U)$
$\Rightarrow$ $\text{pre-}i^{-1}\mathcal{G}$ already sheaf, $\mathcal{G}|_X = i^{-1}\mathcal{G}$ same as def. last time.

Exercise $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)} \ \forall p \in X$.

Adjoint property: $f: X \to Y$ cont., $\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on Y.
Let $\varphi: \mathcal{G} \to f_*\mathcal{F}$ morphism of sheaves on Y.
If $U \subseteq X$ open, $V \supseteq f(U)$, then
$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$$
Compatible with restriction maps on $\mathcal{G}$, so induce:
$$\psi_U : \text{pre-}f^{-1}\mathcal{G}(U) \longrightarrow \mathcal{F}(U).$$
This gives morphism ψ : pre-$f^{-1}\mathcal{G} \to \mathcal{F}$, inducing $\psi^+ : f^{-1}\mathcal{G} \to \mathcal{F}$.
Exercise $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$, $\varphi \mapsto \psi^+$ iso. of abelian gps.
Category Theory: f^{-1} left adjoint to f_*, f_* right adj. to f^{-1}.

X variety (ringed space).
Def An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ on X s.t. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module $\forall U \subseteq X$ open, and if $V \subseteq U$ then $\mathcal{F}(U) \to \mathcal{F}(V)$ is an $\mathcal{O}_X(U)$-homomorphism (i.e. $f \in \mathcal{O}_X(U)$, $\sigma \in \mathcal{F}(U) \Rightarrow (f\cdot\sigma)|_V = f|_V \cdot \sigma|_V$.)
An $\mathcal{O}_X$-homomorphism $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves s.t. $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-hom. $\forall U \subseteq X$ open.
Note: 1) $\ker(\varphi)$, $\mathrm{Im}(\varphi)$ are $\mathcal{O}_X$-modules.
2) $\mathcal{F}$ $\mathcal{O}_X$-module $\Rightarrow$ $\mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module
Def $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = [U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)]^+$.
Exercise $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{G}_p$.
Def $\mathcal{F}$ $\mathcal{O}_X$-module. $\mathcal{F}$ is locally free of rank r, if $\exists X = \bigcup_\alpha U_\alpha$ s.t. $\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ $\forall \alpha$. A locally free sheaf of rank one is invertible.
Example $\pi: \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$. Let $m \in \mathbb{Z}$.
Def. sheaf $\mathcal{O}(m) = \mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$ by
$$\Gamma(U, \mathcal{O}(m)) = \{ h \in k[\pi^{-1}(U)] \mid h(\lambda x) = \lambda^m h(x) \ \forall x \in \pi^{-1}(U),\ \lambda \in k^* \}.$$
Note: $f \in S = k[x_0, .., x_n]$ form of deg. >0.
$\Rightarrow k[\pi^{-1}(D_+(f))] = k[D(f)] = S_f$ graded ring. $\Gamma(D_+(f), \mathcal{O}(m)) = (S_f)_m$.

$\mathcal{O}(m)$ is an $\mathcal{O}_{\mathbb{P}^n}$-module: If $f \in \mathcal{O}_{\mathbb{P}^n}(U)$, $h \in \Gamma(U, \mathcal{O}(m))$, set
$$f \cdot h = (f \circ \pi) \cdot h \in k[\pi^{-1}(U)].$$

Notice: $\mathcal{O}(m)$ invertible sheaf.

On $U_i = D_+(x_i) \subseteq \mathbb{P}^n$ we have $\mathcal{O}_{U_i} \xrightarrow{\cong} \mathcal{O}(m)|_{U_i}$, $h \mapsto x_i^m h$.

Note also: $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$.
$$\Gamma(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} S_m, & m \geq 0 \\ 0, & m < 0. \end{cases}$$

Lemma $\mathcal{O}(m) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(p) \cong \mathcal{O}(m+p)$.

Proof

$U \subseteq \mathbb{P}^n$ open: $\Gamma(U, \mathcal{O}(m)) \otimes \Gamma(U, \mathcal{O}(p)) \longrightarrow \Gamma(U, \mathcal{O}(m+p))$
$$f \otimes g \longmapsto f \cdot g$$

Induces $\mathcal{O}(m) \otimes \mathcal{O}(p) \longrightarrow \mathcal{O}(m+p)$.

Restrict to U_i: $\mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}$.

□

Consequence: $\mathcal{O}(m) \cong \mathcal{O}(p) \iff m = p$.

Later: Any inv. sheaf on $\mathbb{P}^n$ is $\cong \mathcal{O}(m)$, some m.

$\mathrm{Pic}(\mathbb{P}^n) = \{\mathcal{O}(m) \mid m \in \mathbb{Z}\}$ Picard group of $\mathbb{P}^n$.

Coherent Sheaves.

X affine variety, $A = k[X]$, M A-module.

Def $\tilde{M} = [U \mapsto M \otimes_A \mathcal{O}_X(U)]^+$ "quasi-coherent" $\mathcal{O}_X$-module.

Note: $M \otimes_A \mathcal{O}_X(D(f)) = M \otimes_A A_f = M_f$, $f \in A$.

Examples 1) $\tilde{A} = \mathcal{O}_X$.

2) $Y \subseteq X$ closed, $I = I(Y) \subseteq A$. Then $\mathcal{I}_Y = \tilde{I}$ and $i_* \mathcal{O}_Y = \widetilde{A/I}$.

Claim: $\tilde{M}_p = M_{I(p)}$ for $p \in X$, $I(p) = I(\{p\}) \subseteq A$.

$M_{I(p)} \longrightarrow \tilde{M}_p$, $m/f \longmapsto (m \otimes 1/f)_p$, $m \otimes 1/f \in \tilde{M}(D(f))$.

Surjective: clear.

Injective: If $m/f \mapsto 0$ then $(m \otimes 1/f)|_{D(h)} = 0$ for some $h \in A \setminus I(p)$.
$$\Rightarrow m \otimes 1/f = 0 \in M \otimes_A A_h = M_h$$
$$\Rightarrow h^N \cdot m = 0 \in M \Rightarrow m/f = 0 \in M_{I(p)}.$$

$\therefore$ $\tilde{M}(U)$ = set of fcns. $s : U \longrightarrow \coprod_{p \in U} M_{I(p)}$ satisfying:

$\forall p \in U \ \exists\, p \in V \subseteq U$ and $m \in M$, $f \in A$ s.t. $s(q) = m/f \in M_{I(q)}$ $\forall q \in V$.

Coherent sheaves X affine variety, $A = k[X]$, M A-module.

Def $\tilde{M} = [U \mapsto M \otimes_A \mathcal{O}_X(U)]^+$ "quasi-coherent" $\mathcal{O}_X$-module.

Note: $M \otimes_A \mathcal{O}_X(D(f)) = M \otimes_A A_f = M_f$, $f \in A$.

Examples 1) $\tilde{A} = \mathcal{O}_X$

2) $Y \subseteq X$ closed, $I = I(Y) \subseteq A \Rightarrow \mathcal{I}_Y = \tilde{I} \subseteq \mathcal{O}_X$ and $i_* \mathcal{O}_Y = \widetilde{A/I}$

Claim: $\tilde{M}_p = M_{I(p)}$ $\forall p \in X$, $I(p) = I(\{p\}) \subseteq A$.

Have map $M_{I(p)} \longrightarrow \tilde{M}_p$, $m/f \longmapsto (\underset{\tilde{M}(D(f))}{\underset{\cap}{m \otimes 1/f}})_p$

Surjective: clear.

Injective: If $m/f \mapsto 0$ then $(m \otimes 1/f)|_{D(h)} = 0$ for some $h \in A \setminus I(p)$

$\Rightarrow m \otimes 1/f = 0 \in M \otimes_A A_h = M_h \Rightarrow h^N \cdot m = 0 \in M \Rightarrow m/f = 0 \in M_{I(p)}$

Consequence:

$\tilde{M}(U) =$ set of functions $s : U \longrightarrow \coprod_{p \in U} M_{I(p)}$ satisfying:

$\forall p \in U$ $\exists$ $p \in V \subseteq U$ and $m \in M$, $f \in A$ s.t. $s(q) = m/f \in M_{I(q)}$ $\forall q \in V$.

Prop $\tilde{M}(D(f)) \cong M_f$. (Note $\Rightarrow \mathcal{O}_X(D(f)) = A_f$.)

Proof

$\varphi : M_f = M \otimes_A \mathcal{O}_X(D(f)) \longrightarrow \tilde{M}(D(f))$.

Injective: If $\varphi(m/f^n) = 0$ then $m/f^n = 0 \in M_{I(p)}$ $\forall$ $p \in D(f)$

$\Rightarrow \forall$ $p \in D(f)$ $\exists$ $h_p \in A \setminus I(p)$: $h_p \cdot m = 0 \in M$.

$D(f) \subseteq \bigcup D(h_p) \Rightarrow V(f) \supseteq V(\{h_p\}) \Rightarrow f \in I(V(\{h_p\}))$

$\Rightarrow f^N = \sum a_p h_p$, $a_p \in A$.

$f^N \cdot m = \sum a_p h_p m = 0 \Rightarrow m/f^n = 0 \in M_f$.

Surjective: $s \in \tilde{M}(D(f))$, $s : D(f) \longrightarrow \coprod M_{I(p)}$

$\exists$ cover $D(f) = \bigcup_{i=1}^{n} V_i$, $m_i \in M$, $h_i \in A$ s.t. $s = m_i/h_i$ on V.

WLOG: $V_i = D(g_i)$

WLOG: $V_i = D(h_i)$ (replace $m_i \mapsto g_i m_i$, $h_i \mapsto g_i h_i$)

On $D(h_i h_j) = D(h_i) \cap D(h_j)$ we have $s = m_i/h_i = m_j/h_j \in \tilde{M}(D(h_i h_j))$

Injectivity for $D(h_i h_j) \Rightarrow m_i/h_i = m_j/h_j \in M_{h_i h_j}$.

$\Rightarrow (h_i h_j)^n (h_j m_i - h_i m_j) = 0$ for some $n > 0$.

Replace m_i with $h_i^n m_i$, h_i with h_i^{n+1}.

WLOG $h_j m_i = h_i m_j \in M \quad \forall i, j$.

$D(f) \subseteq \bigcup D(h_i) \Rightarrow f^N = \sum a_i h_i$, $a_i \in A$.

Set $m = \sum a_i m_i \in M$.

<u>Claim</u>: $s = \psi(m/f^N)$.

$\forall j: \quad h_j \cdot m = \sum_i h_j a_i m_i = \sum_i a_i h_i m_j = f^N m_j$

$\Rightarrow m/f^N = m_j/h_j = s$ on $D(h_j)$

$\therefore$ $s = m/f^N$ everywhere.

$\square$

<u>Note</u>: $\Gamma(X, \tilde{M}) = M$.

<u>Def</u> X any variety. An $\mathcal{O}_X$-module $\mathcal{F}$ is <u>quasi-coherent</u> if $\exists$ open affine cover $X = \bigcup U_i$ and $k[U_i]$-modules M_i s.t. $\mathcal{F}|_{U_i} \cong \tilde{M}_i \quad \forall i$.

$\mathcal{F}$ is <u>coherent</u> if M_i is a f.g. $k[U_i]$-module $\forall i$.

<u>Examples</u> 1) locally free $\mathcal{O}_X$-modules are coherent.

2) $Y \subseteq X$ closed $\Rightarrow$ $\mathcal{I}_Y$ and $i_* \mathcal{O}_Y$ are coherent.

<u>Example</u> $X = \mathbb{A}^1$, $\mathcal{F}(U) = \begin{cases} \mathcal{O}_{\mathbb{A}^1}(U) & \text{if } 0 \notin U \\ 0 & \text{if } 0 \in U. \end{cases}$ "extension of $\mathcal{O}_{\mathbb{A}^1 \setminus 0}$ by zero".

$\mathcal{F}$ is an $\mathcal{O}_X$-module, but NOT quasi-coherent:

If $U \subseteq \mathbb{A}^1$ open affine, $0 \in U$, then $\mathcal{F}(U) = 0$ but $\mathcal{F}|_U \neq 0 = \tilde{0}$.

<u>Exercises</u> 1) $\mathcal{F}$ quasi-coherent $\Leftrightarrow$ $\mathcal{F}|_U = \widetilde{\mathcal{F}(U)} \quad \forall U \subseteq X$ open affine.

2) $\mathcal{F}$ coherent $\Rightarrow$ $\mathcal{F}(U)$ f.g. $k[U]$-module $\forall U \subseteq X$ affine.

<u>Exercise</u> $f: X \to Y$ morphism of varieties.

1) f affine $\Rightarrow$ $f_* \mathcal{O}_X$ quasi-coh. $\mathcal{O}_Y$-module.

2) f finite $\Rightarrow$ $f_* \mathcal{O}_X$ coherent $\mathcal{O}_Y$-module.

Example M f.g. A-module, $X = \operatorname{Specm}(A)$.

Then $\tilde{M}$ locally free $\mathcal{O}_X$-module of rank r $\Leftrightarrow$

M projective A-module of constant rank r.

($\tilde{M}$ loc. free $\Leftrightarrow X = \bigcup D(f_i)$, $\tilde{M}|_{D(f_i)} \cong \mathcal{O}_{D(f_i)}^{\oplus r}$ $\forall i$

$\Leftrightarrow M_{f_i} \cong A_{f_i}^{\oplus r}$ $\forall i$

$\Leftrightarrow M$ projective.)

Recall: X complete $\Rightarrow \Gamma(X, \mathcal{O}_X) = k$

More general Fact: $\mathcal{F}$ coherent sheaf on complete var. $X \Rightarrow \dim_k \Gamma(X, \mathcal{F}) < \infty$.

Note: $\Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = k[t]$, $\dim = \infty$.

Pushforward, pullback

$f: X \longrightarrow Y$ morphism of varieties (SWFs).

$\mathcal{F}$ $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module.

Have ring hom. $f^*: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \Rightarrow$ $f_*\mathcal{F}$ also $\mathcal{O}_Y$-module.

$\mathcal{G}$ $\mathcal{O}_Y$-module. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module $(f^{-1}h \cdot f^{-1}s = f^{-1}(hs).)$

$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ corresponds to $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

Def $f^*\mathcal{G} = (f^{-1}\mathcal{G}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ "pullback of $\mathcal{G}$ to X by f."

Examples 1) $f^*\mathcal{O}_Y = \mathcal{O}_X$.

2) $(f^*\mathcal{G})_p = (f^{-1}\mathcal{G})_p \otimes_{(f^{-1}\mathcal{O}_Y)_p} \mathcal{O}_{X,p} = \mathcal{G}_{f(p)} \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p}$

3) $U \subseteq Y$ open, $i: U \longrightarrow Y$. $i^*\mathcal{G} = \mathcal{G}|_U \otimes_{\mathcal{O}_Y|_U} \mathcal{O}_U = \mathcal{G}|_U$.

Adjoint property

$f: X \longrightarrow Y$ morphism, $\mathcal{G}$ $\mathcal{O}_Y$-module.

Have $f^{-1}\mathcal{O}_Y$-~~morphism~~ homomorphism: $f^{-1}\mathcal{G} \longrightarrow f^*\mathcal{G}$, $s \longmapsto s \otimes 1$.

Gives $\mathcal{O}_Y$-homomorphism $\alpha: \mathcal{G} \longrightarrow f_*f^*\mathcal{G}$

Lemma $\mathcal{F}$ $\mathcal{O}_X$-module, $\mathcal{G}$ $\mathcal{O}_Y$-module.

Then $\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$

<u>Proof</u> Given $\psi : f^*\mathcal{G} \longrightarrow \mathcal{F}$, obtain $\varphi : \mathcal{G} \xrightarrow{\alpha} f_* f^* \mathcal{G} \xrightarrow{f_*\psi} f_*\mathcal{F}$

Given $\varphi : \mathcal{G} \longrightarrow f_*\mathcal{F}$, obtain $\tilde{\varphi} : f^{-1}\mathcal{G} \longrightarrow \mathcal{F}$ $\quad f^{-1}\mathcal{O}_Y$-hom.

Take $\psi : f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow \mathcal{F}$, $\quad s \otimes h \longmapsto h \cdot \tilde{\varphi}(s)$.

□

<u>Functoriality</u> $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphisms.

$\mathcal{F}$ sheaf on $X \Rightarrow (gf)_*\mathcal{F} = g_*(f_*\mathcal{F})$.

<u>Prop</u> 1) $\mathcal{G}$ sheaf on $Z \Rightarrow (gf)^{-1}\mathcal{G} = f^{-1}(g^{-1}\mathcal{G})$

2) $\mathcal{G}$ $\mathcal{O}_Z$-module $\Rightarrow (gf)^*\mathcal{G} = f^*(g^*\mathcal{G})$.

<u>Proof</u> 2)

$\mathrm{id} : (gf)^*\mathcal{G} \longrightarrow (gf)^*\mathcal{G}$ gives

$\mathcal{G} \longrightarrow (gf)_*(gf)^* = g_* f_* (gf)^*\mathcal{G}$ gives

$g^*\mathcal{G} \longrightarrow f_*(gf)^*\mathcal{G}$ gives $f^* g^* \mathcal{G} \longrightarrow (gf)^*\mathcal{G}$

Have global morphism, check stalks!

$$f^*(g^*\mathcal{G})_p = (g^*\mathcal{G})_{f(p)} \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p} = \left(\mathcal{G}_{gf(p)} \otimes_{\mathcal{O}_{Z,gf(p)}} \mathcal{O}_{Y,f(p)}\right) \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p}$$

$$= \mathcal{G}_{gf(p)} \otimes_{\mathcal{O}_{Z,gf(p)}} \mathcal{O}_{X,p} = ((gf)^*\mathcal{G})_p .$$

□

<u>Consequence</u>: $\mathcal{G}$ locally free $\mathcal{O}_Y$-module $\Rightarrow$
$f^*\mathcal{G}$ locally free $\mathcal{O}_X$-module.

$\boxed{\text{Sections: If } \sigma \in \mathcal{G}(V)\text{, set } f^*\sigma = f^{-1}\sigma \otimes 1 \in \Gamma(f^{-1}(V), f^*\mathcal{G}).}$

$f: X \to Y$ morphism of ~~varieties~~ SWFS. $\mathcal{G}$ $\mathcal{O}_Y$-module.

$f^{-1}\mathcal{G}$ $f^{-1}\mathcal{O}_Y$-module, have $f^*: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

Def $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

$\boxed{\text{Stalk: } (f^*\mathcal{G})_p = \mathcal{G}_{f(p)} \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p}.}$

If $Z \xrightarrow{g} X \xrightarrow{f} Y$ then $(fg)^*\mathcal{G} = g^*(f^*\mathcal{G})$

Consequence: $\mathcal{G}$ locally free $\mathcal{O}_Y$-module $\Rightarrow$ $f^*\mathcal{G}$ locally free $\mathcal{O}_X$-mod

Proof: $Y = \bigcup V_\alpha$, $\mathcal{G}|_{V_\alpha} \cong \mathcal{O}_{V_\alpha}^{\oplus r}$.

Set $U_\alpha = f^{-1}(V_\alpha) \subseteq X$. Then $X = \bigcup_\alpha U_\alpha$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U_\alpha \arrow[u, "i"] \arrow[r, "f'"] & V_\alpha \arrow[u, "j"'] \end{tikzcd}$$

$(f^*\mathcal{G})|_{U_\alpha} = i^* f^* \mathcal{G} = f'^* j^* \mathcal{G} = f'^*(\mathcal{G}|_{V_\alpha})$

$\cong f'^*(\mathcal{O}_{V_\alpha}^{\oplus r}) = \mathcal{O}_{U_\alpha}^{\oplus r}$.

□

Lemma $f: X \to Y$ morphism, $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ $\mathcal{O}_Y$-modules.

Then $f^*\mathcal{G}' \to f^*\mathcal{G} \to f^*\mathcal{G}'' \to 0$ exact on X.

$\mathcal{G}''$ locally free $\Rightarrow$ first map is injective.

Proof Let $p \in X$. Stalks of seq. on X:

$$\mathcal{G}'_{f(p)} \otimes \mathcal{O}_{X,p} \longrightarrow \mathcal{G}_{f(p)} \otimes \mathcal{O}_{X,p} \longrightarrow \mathcal{G}''_{f(p)} \otimes \mathcal{O}_{X,p} \longrightarrow 0$$

$\boxed{\otimes \text{ over } \mathcal{O}_{Y,f(p)}}$

is exact because $0 \to \mathcal{G}'_{f(p)} \to \mathcal{G}_{f(p)} \to \mathcal{G}''_{f(p)} \to 0$ is exact, $\otimes \mathcal{O}_{X,p}$ right exact.

$\mathcal{G}''$ loc. free $\Rightarrow$ $\mathcal{G}''_{f(p)}$ free $\mathcal{O}_{Y,f(p)}$-module $\Rightarrow$ split exact.

□

Def The $\mathcal{O}_X$-module $\mathcal{F}$ is generated by (fin. many) global sections if $\exists\, \mathcal{O}_X^{\oplus m} \twoheadrightarrow \mathcal{F}$

Equivalent: $\exists\, s_1, \dots, s_m \in \Gamma(X, \mathcal{F})$ s.t. $\mathcal{F}_p$ gen by $(s_1)_p, \dots, (s_m)_p$ as $\mathcal{O}_{X,p}$-module $\forall p$.

Example $\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Example Any quasi-coherent sheaf on an affine variety.

Suppose $f: X \to \mathbb{P}^n$ morphism.

$\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow \mathcal{O}(1) \longrightarrow 0 \Rightarrow \mathcal{O}_X^{\oplus n+1} \longrightarrow f^*\mathcal{O}(1) \longrightarrow 0$ on X.

$\Rightarrow f^*\mathcal{O}(1)$ gen. by $f^*(x_0), \dots, f^*(x_n) \in \Gamma(X, f^*\mathcal{O}(1))$.

Prop X variety, $\mathcal{L}$ invertible sheaf gen. by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$.
Then $\exists!\ f: X \to \mathbb{P}^n$ s.t. $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*(x_i)\ \forall i$.

Proof Set $U_i = \{p \in X \mid (s_i)_p \notin m_p \mathcal{L}_p\} \subseteq X$.

U_i is open: If $V \subseteq X$ open s.t. $\mathcal{L}|_V \cong \mathcal{O}_V$, $t \in \mathcal{L}(V)$ gen. $\mathcal{L}|_V$,
then write $s_i = h_i \cdot t$ on V, $h_i \in \mathcal{O}_X(V)$.
Then $U_i \cap V = \{p \in V \mid h_i \notin m_p\} = \{p \in V \mid h_i(p) \neq 0\}$ open in V.

Note: 1) $\mathcal{L}$ gen by $s_0, \dots, s_n \Rightarrow X = \bigcup_{i=0}^n U_i$

2) $\mathcal{O}_{U_i} \xrightarrow{\cong} \mathcal{L}|_{U_i}$, $1 \mapsto s_i$. "s_i generates $\mathcal{L}$ on U_i"

Write $s_j = h_{ij} \cdot s_i$ on U_i, $h_{ij} \in k[U_i]$.

Def $f: U_i \to \mathbb{P}^n$, $f(p) = (h_{i0}(p) : \dots : h_{in}(p))$

Compatible: On $U_\ell \cap U_i$ we have $h_{\ell j} s_\ell = s_j = h_{ij} s_i = h_{ij} h_{\ell i} s_\ell$
$\Rightarrow h_{\ell j} = h_{\ell i} \cdot h_{ij}$

Map on U_ℓ: $p \mapsto (h_{\ell 0}(p) : \dots : h_{\ell n}(p)) = (h_{\ell i}(p) \cdot h_{i0}(p) : \dots : h_{\ell i}(p) h_{in}(p))$ same!

$\therefore$ Have morphism $f: X \to \mathbb{P}^n$.

Claim: $\exists$ iso $\mathcal{L} \xrightarrow{\sim} f^*\mathcal{O}(1)$, $s_i \mapsto f^*(x_i)$

On U_i, def. $\mathcal{L}|_{U_i} \xrightarrow{\sim} f^*\mathcal{O}(1)$, $h \cdot s_i \mapsto h \cdot f^*(x_i)$.

This means $s_\ell = h_{i\ell} s_i \mapsto h_{i\ell} f^*(x_i)$ on U_i.

Must show: $h_{i\ell} \cdot f^*(x_i) = f^*(x_\ell)$ on U_i.

Def of $f \Rightarrow (x_\ell / x_i) \circ f = h_{i\ell} / h_{ii} = h_{i\ell} \in k[U_i]$

$\Rightarrow f^*(x_\ell) = f^*(\frac{x_\ell}{x_i} \cdot x_i) = f^*(x_\ell / x_i) \cdot f^*(x_i) = h_{i\ell} \cdot f^*(x_i)$.

Uniqueness: $f: X \to \mathbb{P}^n$ any morphism s.t. $\mathcal{L} \cong f^*\mathcal{O}(1)$, $s_i \leftrightarrow f^*(x_i)$.

Then on U_i: $h_{i\ell} s_i = s_\ell = f^*(x_\ell) = f^*(\frac{x_\ell}{x_i} \cdot x_i) = f^*(x_\ell/x_i) \cdot f^*(x_i) = f^*(x_\ell/x_i)\, s_i$

$\Rightarrow h_{i\ell} = (x_\ell/x_i) \circ f \quad \forall i, \ell$.

$\therefore$ f must be defined as above.

$\square$

Def $\mathcal{L}$ is a very ample invertible sheaf on X is a basis $s_0, \dots, s_n$ of $\Gamma(X, \mathcal{L})$ gives an iso. $f: X \xrightarrow{\sim} W \subseteq \mathbb{P}^n$, where $W \subseteq \mathbb{P}^n$ locally closed.

Ex: $\mathcal{O}_{\mathbb{P}^n}(m)$ very ample $\Leftrightarrow m \geq 1$.

Def $PGL(n) = GL(n+1)/k^*$

Exercise $PGL(n)$ is an affine alg. group.

Fact: Every invertible sheaf on $\mathbb{P}^n$ is $\cong \mathcal{O}(m)$, $m \in \mathbb{Z}$.

Conséquence $\mathrm{Aut}(\mathbb{P}^n) = PGL(n)$.

Proof Clear that $PGL(n) \subseteq \mathrm{Aut}(\mathbb{P}^n)$.

Let $f: \mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$ automorphism.

FACT $\Rightarrow$ $f^*\mathcal{O}(1) \cong \mathcal{O}(m)$

$\binom{n+m}{m} = \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}(m)) = \dim_k \Gamma(\mathbb{P}^n, f^*\mathcal{O}(1)) = \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}(1)) = n+1$

$\Rightarrow m = 1$.

Write $f^*(x_i) = \sum_{j=0}^{n} a_{ij} x_j \in \Gamma(\mathcal{O}(1))$, $a_{ij} \in k$.

Set $A = (a_{ij})$ ~~[illegible]~~ $\in GL(n+1)$.

Def $\varphi : \mathbb{P}^n \to \mathbb{P}^n$, $\varphi(x_0 : \cdots : x_n) = (\sum a_{0j} x_j : \cdots : \sum a_{nj} x_j)$

$\varphi^*(x_\ell / x_i) = (\sum a_{\ell j} x_j)/(\sum a_{ij} x_j) = f^*(x_\ell / x_i) \Rightarrow f = \varphi \in PGL(n)$.

$\square$

Cor $\mathbb{P}^n$ not alg. group for $n \geq 1$.

Normal varieties

Def X irred. variety. X is normal if $\mathcal{O}_{X,p}$ normal $\forall p \in X$.

Example: non-sing. varieties.

Note: X affine: X normal $\Leftrightarrow$ $k[X]_M$ normal $\forall M \subseteq k[X]$ max ideal

$\Leftrightarrow$ $k[X]$ normal.

Exercise A domain, $S \subseteq A$ mult. subset. Then $\overline{S^{-1}A} = S^{-1}\bar{A}$.

Lemma $\varphi : U \to X$ morphism of affine varieties.

φ open embedding $\Leftrightarrow$ $\exists f_1, \dots, f_n \in k[X]$ s.t. $(\varphi^* f_1, \dots, \varphi^* f_n) = (1) = k[U]$ and $\varphi^* : k[X]_{f_i} \xrightarrow{\sim} k[U]_{\varphi^* f_i}$ iso. $\forall i$.

Proof

$\Rightarrow$: Take open cover $U = \bigcup D(f_i)$, $f_i \in k[X]$.

$\Leftarrow$: Set $V = \bigcup D(f_i) \subseteq X$.

$(\varphi^* f_1, \dots, \varphi^* f_n) = k[U] \Rightarrow \varphi(U) \subseteq V$.

Assumptions $\Rightarrow$ $\varphi : \varphi^{-1}(D(f_i)) \xrightarrow{\sim} D(f_i)$ iso $\forall i$

$\therefore$ $\varphi : U \xrightarrow{\sim} V$.

$\square$

Def X irred. affine variety. $\bar{X} = \mathrm{Specm}(\overline{k[X]})$ normalization of X.

Lemma Let X be affine, $U \subseteq X$ open affine. Then $\bar{U} \subseteq \bar{X}$ open subvar.

Proof Since $k[X] \subseteq k[U] \subseteq k(X)$ we get $\overline{k[X]} \subseteq \overline{k[U]}$, morphism $\varphi : \bar{U} \to \bar{X}$.

Take $f_1, \dots, f_n \in k[X]$ as in Lemma wrt. $U \subseteq X$.

Then $(f_1, \dots, f_n) = (1) = k[U]$. And $\overline{k[U]_{f_i}} = \overline{k[U]_{f_i}} = \overline{k[X]_{f_i}} = \overline{k[X]}_{f_i}$. Lemma $\Rightarrow$ φ open emb. $\square$

Construction

X irred. variety. $X = X_1 \cup \dots \cup X_n$ open affine cover.

Want to glue $\overline{X_1}, \dots, \overline{X_n}$ to normal variety $\overline{X}$.

Set $U_{ij} = X_i \cap X_j \subseteq X_i$

Then U_{ij} affine.

Have iso. $\phi_{ij} = id : U_{ij} \xrightarrow{\simeq} U_{ji}$, and $\phi_{ij} = \phi_{kj} \circ \phi_{ik}$ $\forall i,j,k$.

Now $\overline{U_{ij}} \subseteq \overline{X_i}$ open, and still have iso. $\phi_{ij} : \overline{U_{ij}} \to \overline{U_{ji}}$,

s.t. $\phi_{ij} = \phi_{kj} \circ \phi_{ik}$.

Can glue the $\overline{X_i}$ by identifying $\overline{U_{ij}} \subseteq \overline{X_i}$ with $\overline{U_{ji}} \subseteq \overline{X_j}$.

Result: pre-variety $\overline{X} = \bigcup \overline{X_i}$ <u>Normalization</u> of X.

<u>Exercise</u>: $\overline{X}$ is separated.

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Lemma $\varphi : U \to X$ morphism of affine varieties.
φ is open embedding $\Leftrightarrow$ $\exists f_1, .., f_n \in k[X]$ s.t. $(\varphi^* f_1, .., \varphi^* f_n) = (1) = k[U]$
and $\varphi^* : k[X]_{f_i} \xrightarrow{\sim} k[U]_{\varphi^* f_i}$ iso $\forall i$.

Proof
$\Rightarrow$: Take open cover $U = \bigcup_{i=1}^{n} D(f_i)$, $f_i \in k[X]$.
$\Leftarrow$: Set $V = \bigcup D(f_i) \subseteq X$.
$(\varphi^* f_1, .., \varphi^* f_i) = (1) \subseteq k[U] \Rightarrow \varphi(U) \subseteq V$.
Assumptions $\Rightarrow$ $\varphi : \varphi^{-1}(D(f_i)) \xrightarrow{\cong} D(f_i)$ iso $\forall i$
$\therefore$ $\varphi : U \xrightarrow{\cong} V$ iso
□

Def X irred. affine variety. Normalization $\overline{X} = \text{spec-m}(\overline{k[X]})$.

Lemma X affine, $U \subseteq X$ open affine $\Rightarrow$ $\overline{U} \subseteq \overline{X}$ open affine.
Pf $k[X] \subseteq k[U] \subseteq k(X)$ $\Rightarrow$ $\overline{k[X]} \subseteq \overline{k[U]}$ $\Rightarrow$ have $\varphi : \overline{U} \to \overline{X}$.
Take $f_1, .., f_n \in k[X]$ as in Lemma wrt. $U \subseteq X$.
Then $(f_1, .., f_n) = (1) \subseteq k[\overline{U}] = \overline{k[U]}$.
And $k[\overline{U}]_{f_i} = \overline{k[U]_{f_i}} = \overline{k[X]_{f_i}} = k[\overline{X}]_{f_i}$
Lemma $\Rightarrow$ φ open embedding.
□

Exercise Given pre-varieties $X_1, .., X_n$, open subsets $U_{ij} \subseteq X_i$, and isos $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ s.t.
1) $U_{ii} = X_i$, $\phi_{ii} = id$.
2) $\forall i,j,k$: $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\phi_{ik} = \phi_{jk}\phi_{ij}$ on $U_{ij} \cap U_{ik}$.

Then $\exists!$ pre-variety X with morphisms $\psi_i : X_i \to X$ s.t.
a) $\psi_i : X_i \xrightarrow{\cong}$ open $\subseteq X$
b) $X = \bigcup \psi(X_i)$
c) $\psi_i(U_{ij}) = \psi_i(X_i) \cup \psi_j(X_j)$
d) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.

Def X normal $\Leftrightarrow$ X irred. and $\mathcal{O}_{X,p}$ normal $\forall p \in X$.

Construction X irred. variety.
$X = X_1 \cup X_2 \cup \dots \cup X_n$ open affine cover.
Set $U_{ij} = X_i \cap X_j$. Then U_{ij} is affine.

Have $\phi_{ij} = id : U_{ij} \xrightarrow{\sim} U_{ji}$, sat (1) + (2).

Now $\overline{U_{ij}} \subseteq \overline{X_i}$ open subset.

Still have $\phi_{ij} : \overline{U_{ij}} \xrightarrow{\sim} \overline{U_{ji}}$ sat. (1) + (2).

Exercise $\Rightarrow$ $\exists$ pre-variety $\overline{X} = \overline{X_1} \cup \overline{X_2} \cup \cdots \cup \overline{X_n}$. <u>Normalization of X.</u>

<u>Note:</u> $k[\overline{X_i}]$ f.g. $k[X_i]$-module, so have finite $\pi : \overline{X_i} \to X_i$.

Glue to morphism $\pi : \overline{X} \longrightarrow X$.

<u>Exercise</u> $\pi : \overline{X} \to X$ is finite. ($\pi^{-1}(X_i) = \overline{X_i}$).

<u>Exercise</u> $\varphi : X \to Y$ affine morphism of pre-varieties.

Then Y separated $\Rightarrow$ X separated.

$\therefore$ $\overline{X}$ separated normal variety.

<u>Example</u> X irred. curve $\Rightarrow$ $\pi : \overline{X} \to X$ resolution of singularities.

<u>Local ring along subvariety</u>

X variety, $V \subseteq X$ closed irred. subvariety.

<u>Def</u> $\mathcal{O}_{X,V} = \varinjlim_{\substack{U \subseteq X \text{ open} \\ U \cap V \neq \emptyset}} \mathcal{O}_X(U)$

X irreducible : $\mathcal{O}_{X,V} = \{ f \in k(X) : f \text{ def. at } \underline{\text{some}} \text{ point in } V \}$

General case : $U \subseteq X$ affine, $U \cap V \neq \emptyset$. $P = I(U \cap V) \subseteq k[U]$ prime.

$\mathcal{O}_{X,V} = k[U]_P$.

<u>Def</u> X is <u>regular along V</u> if $\mathcal{O}_{X,V}$ regular local, i.e.

max ideal of $\mathcal{O}_{X,V}$ gen. by $\dim(\mathcal{O}_{X,V}) = \operatorname{codim}(V; X)$ elts.

This happens $\Leftrightarrow$ $V \not\subseteq X_{sing}$.

~~<u>Divisors</u>~~

~~X irred. variety.~~

~~<u>Def</u> A <u>prime divisor</u> on X is a closed irred. subvariety $Y \subseteq X$ of codim 1.~~

~~Let $f \in k(X)^*$. Define $v_Y(f) = \text{length}_{\mathcal{O}_{X,Y}}(\mathcal{O}_{X,Y}/(f))$~~

~~<u>Lemma</u> $v_Y(f) = 0$ for all but finitely many prime divisors.~~

~~<u>Proof</u>~~

Divisors X normal variety.

Def A *prime divisor* on X is a closed irred subvar. $Y \subseteq X$ of codim 1.

Note Y prime div $\Rightarrow$ $\mathcal{O}_{X,Y}$ Noeth. local normal domain of dim 1 $\Rightarrow$ $\mathcal{O}_{X,Y}$ DVR.

Consequences 1) $\operatorname{codim}(X_{sing}\,; X) \geq 2$.

2) Have valuation $v_Y : k(X)^* \longrightarrow \mathbb{Z}$ for any prime div $Y \subseteq X$.

Lemma Let $f \in k(X)^*$. Then $v_Y(f) = 0$ for all but finitely many Y.

Pf Will show $v_Y(f) < 0$ for finitely many Y.

Set $U \subseteq X$ open set where f defined, $Z = X \setminus U$.

$v_Y(f) < 0 \Rightarrow f \notin \mathcal{O}_{X,Y} \Rightarrow f$ not. def. on any point of Y

$\Rightarrow Y \subseteq Z$ component.

$\square$

Def $\operatorname{Div}(X)$ = free abelian group gen by prime divisors

Elements: $D = \sum_{\text{finite}} n_i [Y_i]$ *Weil divisor*.

Def For $f \in k(X)^*$, set $(f) = \sum_Y v_Y(f) \cdot [Y] \in \operatorname{Div}(X)$. "principal divisor of f."

Note: $(f^{-1}) = -(f)$

$(fg) = (f) + (g)$.

$\Rightarrow k(X)^* \longrightarrow \operatorname{Div}(X)$ group hom.

Def $\mathrm{Cl}(X) = \operatorname{Div}(X) / \{(f) : f \in k(X)^*\}$.

Example $\mathrm{Cl}(\mathbb{A}^n) = 0$.

> **Remark** If X complex, then $\mathrm{Cl}(X) \leftrightarrow H_2(X;\mathbb{Z})$

Remark If X not normal, can still define $\mathrm{Cl}(X)$, by using

$v_Y(f/g) = \operatorname{length}_{\mathcal{O}_{X,Y}}(\mathcal{O}_{X,Y}/(f)) - \operatorname{length}_{\mathcal{O}_{X,Y}}(\mathcal{O}_{X,Y}/(g))$

for $f, g \in \mathcal{O}_{X,Y}$.

Divisors on $\mathbb{P}^n$

Note: All prime divisors on $\mathbb{P}^n$ are hypersurfaces $V_+(h)$, $h \in k[x_0, \dots, x_n]$ irred. form.

Def $\deg : \operatorname{Div}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$, $\deg\left(\sum m_i [Y_i]\right) = \sum m_i \deg(Y_i)$.

If $f \in k(\mathbb{P}^n)^*$ then $f = \prod_{i=1}^{r} h_i^{m_i}$, where

$h_i \in k[x_0, \dots, x_n]$ irred. form, $m_i \in \mathbb{Z}$, $\sum m_i \deg(h_i) = 0$.

$Y_i = V_+(h_i) \subseteq \mathbb{P}^n$ prime divisor. $v_{Y_i}(f) = m_i$.

$(f) = \sum_{i=1}^{r} m_i [Y_i]$. $\Rightarrow$ $\deg(f) = 0$.

Obtain: $\deg : Cl(\mathbb{P}^n) \longrightarrow \mathbb{Z}$

Claim: $Cl(\mathbb{P}^n) = \mathbb{Z}$.

$H \subseteq \mathbb{P}^n$ hyperplane $\Rightarrow$ $\deg([H]) = 1$, so $\deg : Cl(\mathbb{P}^n) \twoheadrightarrow \mathbb{Z}$.
surjective.

Injective: Let $D = \sum m_i [Y_i]$, suppose $\deg(D) = 0$.

$Y_i = V_+(h_i)$, h_i irred. form.

$\sum m_i \deg(h_i) = \deg(D) = 0 \Rightarrow f = \prod h_i^{m_i} \in k(\mathbb{P}^n)^*$.

$D = (f) \Rightarrow D = 0 \in Cl(\mathbb{P}^n)$.

Later: X non-singular $\Rightarrow$ $Cl(X) \cong Pic(X)$.

$\therefore Pic(\mathbb{P}^n) = \mathbb{Z}$.

Theorem R normal Noetherian domain $\Rightarrow$ $R = \bigcap_{\dim(R_p)=1} R_p$

Consequence:

X normal, $f \in k(X)^*$.

$f \in k[X] \Leftrightarrow v_Y(f) \geq 0 \quad \forall$ prime div. $Y \subseteq X$.

I.e. the closed subset where f not def. is empty or has codim. 1.

X normal variety, $Y \subseteq X$ prime divisor $\Rightarrow$ $\mathcal{O}_{X,Y}$ DVR.

Thm R normal Noetherian domain $\Rightarrow$ $R = \bigcap_{\text{height}(P)=1} R_P$

Def $\text{height}(P) = \dim(R_P) = \max m$ s.t. $\exists\ 0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m = P$.

Consequence X normal, $f \in k(X)^*$. Then $f \in k[X] \Leftrightarrow v_Y(f) \geq 0\ \ \forall Y \subseteq X$ prime div.

Lemma R Noetherian domain. R UFD $\Leftrightarrow$ every prime of height 1 is principal.

Proof $\Rightarrow$: Assume R UFD, $P \subseteq R$ has height 1.
Let $x \in P$ be any irred. element. $0 \subsetneq (x) \subseteq P \Rightarrow (x) = P$.

$\Leftarrow$: Exercise: R Noetherian domain $\Rightarrow$ every element $=$ product of irred. elts.
Unique factorizations: Show: $r \in R$ irred., $r \mid f \cdot g \Rightarrow r \mid f$ or $r \mid g$.
Show: $(r) \subseteq R$ prime ideal.
Let $P \subseteq R$ min. prime s.t. $(r) \subseteq P$.
PIT $\Rightarrow$ $\text{height}(P) = 1 \Rightarrow P = (s)$, $\quad r = as$, $a \in R$.
r irred $\Rightarrow$ a unit, $P = (s) = (r)$.

□

Prop X irred. affine variety. $k[X]$ UFD $\Leftrightarrow$ X normal and $Cl(X) = 0$.

Proof $\Rightarrow$: Any UFD is normal.
If $Y \subseteq X$ prime divisor then $\text{height}(I(Y)) = 1 \Rightarrow I(Y) = (f)$, $f \in k[X]$.
$(f) = [Y] \Rightarrow [Y] = 0 \in Cl(X)$.

$\Leftarrow$: Let $P \subseteq k[X]$ be a prime of height 1. Show P principal.
$Y = V(P) \subseteq X$ prime div.
$[Y] = 0 \in Cl(X) \Rightarrow [Y] = (f)$, $f \in k(X)^*$.
$v_Z(f) \geq 0\ \ \forall \underset{\text{prime div}}{Z \subseteq X} \Rightarrow f \in k[X]$.
Claim: $P = (f) \subseteq k[X]$.
$\supseteq$: clear.
$\subseteq$: Let $g \in P$. $v_Y(g) \geq 1 \Rightarrow v_Z(g/f) \geq 0\ \ \forall Z$
$\Rightarrow a = g/f \in k[X] \Rightarrow g = af \in (f)$.

□

Prop X normal, $Z \subseteq X$ proper closed subset. $U = X \setminus Z$.

(a) $Cl(X) \twoheadrightarrow Cl(U)$, $\quad [Y] \longmapsto \begin{cases} [Y \cap U] & \text{if } Y \cap U \neq \emptyset \\ 0 & \text{else} \end{cases}$

(b) $\text{codim}(Z; X) \geq 2 \Rightarrow Cl(X) = Cl(U)$.

(c) Z prime divisor $\Rightarrow$ $\mathbb{Z} \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0$ exact.
$\quad m \longmapsto m[Z]$

Proof Def. $\mathrm{Div}(X) \longrightarrow \mathrm{Div}(U)$ by $[Y] \longmapsto \begin{cases} [Y \cap U] \\ 0 \end{cases}$

$(f) \mapsto (f|_U)$ for $f \in k(X)^*$ $\Rightarrow$ Well def. group hom. $Cl(X) \longrightarrow Cl(U)$.

(a) If $V \subseteq U$ prime div. then $[V]$ = image of $[\bar{V}]$,

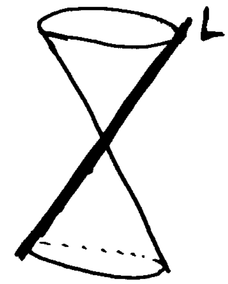
(b) $\mathrm{Div}(X) = \mathrm{Div}(U)$ and $(f) \leftrightarrow (f|_U)$. (Because $\mathcal{O}_{X,Y} = \mathcal{O}_{U,U\cap Y}$ $\forall Y$.)

(c) If $D = \sum n_Y [Y] \longmapsto 0 \in Cl(U)$ then $\exists\, f \in k(X)^*$ s.t.

$v_Y(f) = n_Y \quad \forall\, Y \neq Z$.

$\Rightarrow D - (f) \in \mathbb{Z}\cdot[Z] \quad \Rightarrow \quad D = m\cdot[Z] \in Cl(X)$.

□



Example $X = V(xy - z^2) \subseteq \mathbb{A}^3$ is normal. (Exercise!)

$L = V(y) \cap X = V(y,z)$ prime div. on X.

Max. ideal of $\mathcal{O}_{X,L}$ gen. by z since $y = \frac{z^2}{x} \in \mathcal{O}_{X,L}$.

Set $U = X \setminus L$.

$k[U] = \left(k[x,y,z]/(xy-z^2)\right)_y = k[y,z]_y$ UFD $\Rightarrow Cl(U) = 0$.

$\mathbb{Z} \to Cl(X) \longrightarrow Cl(U) = 0$ exact $\Rightarrow$ $Cl(X)$ gen. by $[L]$.

Note $y \in k(X)^*$. $(y) = v_L(y)\cdot[L] = v_L(z^2)\cdot[L] = 2\cdot[L]$

$\Rightarrow 2\cdot[L] = 0 \in Cl(X)$.

$Cl(X) = \mathbb{Z}/2\mathbb{Z}$ or $\cancel{Cl(X) = 0}$. since $k[X] = k[x,y,z]/(xy-z^2)$ NOT UFD.

Picard Groups

X any variety.

$\mathcal{L}_1, \mathcal{L}_2$ invertible $\mathcal{O}_X$-modules $\Rightarrow$ $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ invertible.

If $\mathcal{L}$ invertible, def. $\mathcal{L}^{-1} = [U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U)]$

Exercise: $\mathcal{L}^{-1}$ inv. $\mathcal{O}_X$-module, and $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Def $\mathrm{Pic}(X) = \{$ iso classes of inv. sheaves on $X \}$ group under $\otimes$.

Notation X irred. variety, $\mathcal{L}$ invertible sheaf, $s \in \mathcal{L}(U)$, $t \in \mathcal{L}(V)$ non-zero sections.

Take $W \subseteq U \cap V$ open s.t. $\mathcal{L}|_W \cong \mathcal{O}_W$, gen. by $u \in \mathcal{L}(W)$.

Then $s|_W = f\cdot u$, $t|_W = g\cdot u$, $f,g \in k[W]$.

Def: $s/t = f/g \in k(X)$. (This is indep. of W and u.)

Example $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Def. $f : X \dashrightarrow \mathbb{A}^n \subseteq \mathbb{P}^n$, $f(x) = (s_1/s_0(x), \dots, s_n/s_0(x))$

If $s_0, \dots, s_n$ generate $\mathcal{L}$, then f extends to morphism $f : X \longrightarrow \mathbb{P}^n$.

<u>More Notation</u> X normal variety, $s \in \mathcal{L}(U)$ non-zero section, $Y \subseteq X$ prime div.

Take $V \subseteq X$ open s.t. $V \cap Y \neq \emptyset$ and $\mathcal{L}|_V \cong \mathcal{O}_V$, gen by $t \in \mathcal{L}(V)$.

<u>Def</u> $v_Y(s) = v_Y(s/t)$.

Well def: If $V' \cap Y \neq \emptyset$, $\mathcal{L}|_{V'}$ gen by $t' \in \mathcal{L}(V')$ then t/t' nowhere zero regular fcn. on $V \cap V'$ $\Rightarrow$ t/t' unit in $\mathcal{O}_{X,Y}$.

$\Rightarrow v_Y(s/t') = v_Y(s/t \cdot t/t') = v_Y(s/t) + 0$.

<u>Def</u> $(s) = \sum_Y v_Y(s)[Y] \in \mathrm{Div}(X)$.

<u>Note</u>: If $s' \in \mathcal{L}(U')$ another non-zero section, then

$(s') = (s) + (s/s')$ $\qquad \Rightarrow \quad (s') = (s) \in \mathcal{Cl}(X)$.

$\therefore$ Have well def. map $\mathrm{Pic}(X) \longrightarrow \mathcal{Cl}(X)$, $\mathcal{L} \mapsto (s)$.

<u>Check</u>: If $s_1 \in \mathcal{L}_1(U)$, $s_2 \in \mathcal{L}_2(U)$ then $s_1 \otimes s_2 \in \mathcal{L}_1 \otimes \mathcal{L}_2(U)$ and

$(s_1 \otimes s_2) = (s_1) + (s_2) \in \mathrm{Div}(X)$.

$\therefore$ $\mathrm{Pic}(X) \longrightarrow \mathcal{Cl}(X)$ group homomorphism.

<u>Cartier Divisors</u> $\boxed{X \text{ normal.}}$

<u>Def</u> A <u>Cartier divisor</u> is a Weil divisor $D = \sum n_i [Y_i]$ which is <u>locally principal</u>.

I.e. $\exists$ open cover $X = \bigcup U_j$ s.t. $D|_{U_j} = 0 \in \mathcal{Cl}(U_j)$

<u>Note</u>: If $D|_{U_j} = (f_j)$, $f_j \in k(U_j)^*$, then we can think of D as the collection $\{f_j\}$ of these generators.

$\rightsquigarrow$ Can define Cartier divs. on any irred. variety.

<u>Def</u> $\mathrm{Ca}\mathcal{Cl}(X) = \{\text{Cartier divisors on } X\} / \{\text{principal divisors}\}$

<u>Note</u>: $\mathcal{L}$ line bundle, $s \in \mathcal{L}(U)$ $\Rightarrow$ $(s) \in \mathrm{Div}(X)$ Cartier div.

$\therefore$ $\mathrm{Pic}(X) \longrightarrow \mathrm{Ca}\mathcal{Cl}(X) \subseteq \mathcal{Cl}(X)$.

<u>Line bundles from divisors</u>

$D = \sum n_Y [Y]$ Weil div. on X.

<u>Def</u> Sheaf $\mathcal{L}(D) = \mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules by

$\Gamma(U, \mathcal{L}(D)) = \{ f \in k(X)^* \mid v_Y(f) \geq -n_Y \ \forall \ Y \text{ prime div.}\} \cup \{0\}$

<u>Example</u> $\mathcal{L}(0) = \mathcal{O}_X(0) = \mathcal{O}_X$.

Example $X = \mathbb{P}^1$, $Q = (a:b) \in \mathbb{P}^1$. $D = n[Q]$.

$Q = V_+(h)$, $h = bx_0 - ax_1 \in k[x_0, x_1]$.

$\mathcal{L}(nQ) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(n)$, $f \mapsto h^n \cdot f$

Note If $h \in k(X)^*$ then $\mathcal{L}(D + (h)) \xrightarrow{\cong} \mathcal{L}(D)$, $f \mapsto h \cdot f$

$(v_Y(f) \geq -n_Y - v_Y(h) \iff v_Y(hf) \geq -n_Y .)$

Consequence: D Cartier divisor $\Rightarrow$ $\mathcal{L}(D)$ line bundle.

(If $D|_U = (h) \in \mathrm{Div}(U)$ $h \in k(U)^*$, then $\mathcal{L}(D)|_U = \mathcal{O}_U((h)) \cong \mathcal{O}_U$.)

Note $\Rightarrow$ have map $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$, $D \mapsto \mathcal{L}(D)$.

Warning If $f \in \Gamma(U, \mathcal{L}(D))$ then f is a rational function.

"(f)" means two things!

Prop $\mathrm{Pic}(X) \cong \mathrm{CaCl}(X)$ as abelian groups.

Proof Must check that maps $\mathrm{Pic}(X) \rightleftarrows \mathrm{CaCl}(X)$ inverse to each other.
– next time.

Examples

1) $\mathrm{Pic}(\mathbb{A}^n) = \mathrm{CaCl}(\mathbb{A}^n) \subseteq \mathrm{Cl}(\mathbb{A}^n) = 0$.

$\Rightarrow$ all line bundles on $\mathbb{A}^n$ are trivial.

(More true: All loc. free sheaves of finite rank are trivial, Serre's conj.)

2) $\mathbb{P}^n = \bigcup_{i=0}^n D_+(x_i)$, and $\mathrm{Cl}(D_+(x_i)) = 0$.

$\Rightarrow$ all Weil divs. are loc. trivial.

$\mathrm{Pic}(\mathbb{P}^n) = \mathrm{CaCl}(\mathbb{P}^n) = \mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$.

Any line bdl. $\cong \mathcal{L}(m[H])$, $m \in \mathbb{Z}$, $H \subseteq \mathbb{P}^n$ hyperplane.

Write $H = V_+(h)$.

$\mathcal{L}(m[H]) \xrightarrow{\cong} \mathcal{O}(m)$, $f \mapsto h^m \cdot f$.

$\therefore \mathrm{Pic}(\mathbb{P}^n) = \{\mathcal{O}(m)\}$.

Cartier divisors $\quad X$ normal.

$D = \sum n_i [Y_i] \in \mathrm{Div}(X)$ is a Cartier divisor if it is locally principal

i.e. $\exists$ open cover $X = \bigcup U_j$ s.t. $D|_{U_j} = 0 \in \mathcal{C}\ell(U_j)$ $\forall j$.

Def $\mathrm{Ca}\mathcal{C}\ell(X) = \{\text{cartier divisors on } X\} / \{\text{principal divisors } (f),\ f \in k(X)^*\}$.

Recall: $\mathcal{L}$ invertible $\mathcal{O}_X$-module, $s \in \mathcal{L}(U)$ non-zero section.

$$(s) = \sum_{Y \subseteq X} v_Y(s)[Y] \in \mathrm{Div}(X).$$

where $v_Y(s) = v_Y(s/t)$, $t \in \mathcal{L}(V)$ local generator, $\mathcal{L}|_V \cong \mathcal{O}_V$, $Y \cap V \neq \emptyset$.

Note: (s) allways Cartier divisor.

$\therefore$ Have group hom. $\mathrm{Pic}(X) \longrightarrow \mathrm{Ca}\mathcal{C}\ell(X) \leq \mathcal{C}\ell(X)$, $\quad \mathcal{L} \mapsto (s)$.

Line bundles from divisors

$D = \sum n_Y [Y]$ Weil divisor on X.

Def Sheaf $\mathcal{L}(D) = \mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules by

$$\Gamma(U, \mathcal{L}(D)) = \{ f \in k(X)^* \mid v_Y(f) \geq -n \ \forall Y \text{ prime div. with } Y \cap U \neq \emptyset \} \cup \{0\}$$

Example $\mathcal{L}(0) = \mathcal{O}_X(0) = \mathcal{O}_X$. $\quad$ $\boxed{\text{since } \mathcal{O}_X(U) = \{f \in k(X)^* \mid f \in \mathcal{O}_{X,Y}\ \forall Y : Y \cap V \neq \emptyset\}}$

Example $X = \mathbb{P}^1$. $Q = (a:b) \in \mathbb{P}^1$. $D = n[Q]$.

$Q = V_+(h)$, $\quad h = bx_0 - ax_1 \in k[x_0, x_1]$.

$\mathcal{L}(nQ) \cong \mathcal{O}_{\mathbb{P}^1}(n)$, $\quad f \mapsto h^n \cdot f$

Note: If $h \in k(X)^*$ then $\mathcal{L}(D + (h)) \xrightarrow{\cong} \mathcal{L}(D)$, $\quad f \mapsto h \cdot f$

$(\ v_Y(f) \geq -n_Y - v_Y(h) \iff v_Y(hf) \geq -n_Y.\ )$

Consequence: D Cartier divisor $\Rightarrow$ $\mathcal{L}(D)$ line bundle.

(If $D|_U = (h) \in \mathrm{Div}(U)$ then $\mathcal{L}(D)|_U = \mathcal{O}_U((h)) \cong \mathcal{O}_U$.)

Note $\Rightarrow$ Have map $\mathrm{Ca}\mathcal{C}\ell(X) \longrightarrow \mathrm{Pic}(X)$, $\quad D \mapsto \mathcal{L}(D)$.

WARNING: If $f \in \Gamma(U, \mathcal{L}(D))$ then f is a rational function.

"(f)" means two things!

Prop $\mathrm{Pic}(X) \cong \mathrm{Ca}\mathcal{C}\ell(X)$ as abelian group.

Proof Must check that $\mathrm{Pic}(X) \rightleftarrows \mathrm{Ca}\mathcal{C}\ell(X)$ are inverse maps.

Let $D = \sum n_Y [Y]$ Cartier div. on X. Set $V = X \setminus \left(\bigcup_{n_Y < 0} Y \right)$.

Then $1 \in k(X)^*$ is a section of $\mathcal{L}(D)$ over V. $\quad (v_Y(1) \geq -n_Y \Leftrightarrow n_Y \geq 0)$.

Claim: $(1) = D$ in $\mathrm{Div}(X)$.

If $D|_U = (h) \in \mathrm{Div}(U)$ then $\mathcal{L}(D)|_U \cong \mathcal{O}_U$, gen. by $h^{-1} \in \Gamma(U, \mathcal{L}(D))$.

$Y \cap U \neq \emptyset \Rightarrow v_Y(1) = v_Y(1/h^{-1}) = v_Y(h)$

$\Rightarrow (1)|_U = D|_U$.

$\therefore$ $\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{CaCl}(X)$ identity.

Let $\mathcal{L}$ line bundle on X, $t \in \mathcal{L}(U)$ non-zero section.

Note: If $0 \neq s \in \mathcal{L}(V)$ then $Y \cap V \neq \emptyset \Rightarrow v_Y(s/t) = v_Y(s) - v_Y(t) \geq -v_Y(t)$

$\Rightarrow s/t \in \Gamma(V, \mathcal{L}((t)))$.

Claim: $\mathcal{L} \longrightarrow \mathcal{L}((t))$, $\quad s \longmapsto s/t$ **isomorphism**.

If $\mathcal{L}|_V = \mathcal{O}_V$, gen. by $u \in \mathcal{L}(V)$, then $(t)|_V = (t/u)|_V$

$\Rightarrow \mathcal{L}((t))|_V$ gen. by u/t.

Since $u \mapsto u/t$, we get $\mathcal{L}|_V \xrightarrow{\cong} \mathcal{L}((t))|_V$.

$\therefore$ $\mathrm{Pic}(X) \longrightarrow \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X)$ identity.

□

Examples

1) $\mathrm{Pic}(\mathbb{A}^n) = \mathrm{CaCl}(\mathbb{A}^n) \subseteq \mathrm{Cl}(\mathbb{A}^n) = 0$.

$\Rightarrow$ All line bundles on $\mathbb{A}^n$ are trivial.

(More is true: All locally free $\mathcal{O}_{\mathbb{A}^n}$-modules of finite rank are trivial.)

2) $\mathbb{P}^n = \bigcup_{i=0}^{n} D_+(x_i)$ and $\mathrm{Cl}(D_+(x_i)) = 0 \Rightarrow$ all Weil divs. are loc. trivial.

$\mathrm{Pic}(\mathbb{P}^n) = \mathrm{CaCl}(\mathbb{P}^n) = \mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$.

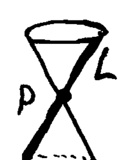
Any line bundle $\cong \mathcal{O}_{\mathbb{P}^n}(m[H])$, $\quad H \subseteq \mathbb{P}^n$ hyper plane.

$H = V_+(h)$, $\quad h \in k[x_0, \dots, x_n]$ linear form.

$\mathcal{O}_{\mathbb{P}^n}(m[H]) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m)$, $\quad f \longmapsto h^m \cdot f$

$\therefore$ $\mathrm{Pic}(\mathbb{P}^n) = \{\mathcal{O}(m)\}$.

3) $X = V(xy - z^2) \subseteq \mathbb{A}^3$



$\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$, gen by $[L]$. $\quad L = V(y) \cap X$.

$I(L) = (y, z) \subseteq k[x, y, z]$.

__Claim__ $[L]$ not Cartier divisor.

otherwise $\exists$ open affine $U \subseteq X$ s.t. $P=(0,0,0) \in U$, and $f \in k(X)^*$ s.t.

$[L \cap U] = (f|_U) \in \mathrm{Div}(U)$.

$\Rightarrow f \in k[U]$ and $I(L\cap U) = (f) \subseteq k[U]$

$\Rightarrow I(L)\cdot \mathcal{O}_{X,P} = (y,z) \subseteq \mathcal{O}_{X,P}$ is principal.

But $P \in X$ singular point $\Rightarrow \dim_k\left(m_P/m_P^2 \right) = 3$; $\{x,y,z\}$ basis.

$\Rightarrow \dim_k\left(\left((y,z)+m_P^2\right)/m_P^2 \right) = 2 \Rightarrow (y,z) \subseteq \mathcal{O}_{X,P}$ __NOT__ principal ↯.

$\therefore\ \mathrm{CaCl}(X) = \mathrm{Pic}(X) = 0 \neq \mathrm{Cl}(X)$.

__Note:__ $\mathcal{O}_{X,P}$ is not a UFD since $I(L)\cdot\mathcal{O}_{X,P}$ has height one, not principal.

__Def__ A irred. variety X is __locally factorial__ if $\mathcal{O}_{X,P}$ UFD $\forall P \in X$.

__Example__ non-singular $\Rightarrow$ locally factorial $\Rightarrow$ normal.

__Prop__ X locally factorial $\Rightarrow \mathrm{Pic}(X) = \mathrm{Cl}(X)$.

__Proof__ Show that every prime divisor $[Y]$ is Cartier.

First: If $U = X \setminus Y$ then $[Y]|_U = 0 \in \mathrm{Div}(U)$ principal.

Let $P \in Y$. $I(Y)\cdot\mathcal{O}_{X,P} \subseteq \mathcal{O}_{X,P}$ height one prime, $\mathcal{O}_{X,P}$ UFD

$\Rightarrow I(Y)\cdot\mathcal{O}_{X,P} = (f) \subseteq \mathcal{O}_{X,P}$, $f \in \mathcal{O}_{X,P} \subseteq k(X)$.

__Note:__ $v_Y(f) = 1$. (further localize.)

If $Z \neq Y$ another prime div, $P \in Z$ then

$f \in \mathcal{O}_{X,Z}$ (def. at P) and

$f \notin I(Z)\cdot\mathcal{O}_{X,P}$ (other height 1 prime)

$\Rightarrow v_Z(f) = 0$.

$\therefore\ (f) = [Y] + \sum n_i [Z_i]$, $P \notin Z_i$.

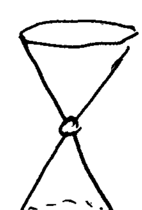
Set $U = X \setminus (\cup Z_i)$.

Then $P \in U$ and $[Y]|_U = (f|_U) \in \mathrm{Div}(U)$ principal.

$\therefore\ [Y]$ is Cartier.

□

__Example__ $X = V(xy - z^2) \subseteq \mathbb{A}^3$, $X_0 = X \setminus \{(0,0,0)\}$



X_0 non-singular $\Rightarrow \mathrm{Pic}(X_0) = \mathrm{Cl}(X_0) = \mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$.

$\exists$ unique non-trivial line bundle on X_0; not restriction of line bdl on X.

$f: X \longrightarrow Y$ morphism of varieties.

Def f is affine if $U \subseteq Y$ open affine $\Rightarrow f^{-1}(U) \subseteq X$ open affine.

f is finite if f affine, and $k[f^{-1}(U)]$ is a finitely gen. $k[U]$-module

Exercise Enough that this holds for open affine cover $Y = \bigcup_i U_i$

Example Any closed embedding $X \hookrightarrow Y$ is finite.

Divisors on Curves.

Recall: X complete non-sing. curve $\Rightarrow$ X is projective. $(X \cong C_K,\ K = k(X))$.

Lemma X complete non-sing curve. Then any non-const. morphism $f: X \to Y$ is finite.

Proof WLOG: Y is a curve (replace Y with closed subset $f(X) \subseteq Y$.)

$f^*: k(Y) \longrightarrow k(X)$ finite field ext.

Take $V \subseteq Y$ open affine. $k[V] \subseteq k(Y)$.

Set $A = \overline{k[V]} \subseteq k(X)$.

Then A f.g. $k[V]$-module.

$U = \mathrm{Specm}(A)$ non-sing. curve.

$k(U) = k(X) \Rightarrow$ have diag.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup\!\!\mid & & \cup\!\!\mid \\ U & \longrightarrow & V \end{array}$$

Claim: $f^{-1}(V) = U$.

$x \in f^{-1}(V) \Rightarrow k[V] \subseteq \mathcal{O}_{X,x} \Rightarrow A \subseteq \mathcal{O}_{X,x} \Rightarrow \mathcal{O}_{X,x} = A_P$, $P \subseteq A$ max ideal.

So $x = P \in \mathrm{specm}(A)$.

$\square$

Def For $f: X \to Y$ a finite morphism, set $\deg(f) = [k(X) : k(Y)]$.

Pullback of divisors on Curves.

$f: X \to Y$ finite morph. of nonsing curves, $Q \in Y$, $\mathfrak{m}_Q = (t) \subseteq \mathcal{O}_{Y,Q}$.

If $f(P) = Q$ then $f^*: \mathcal{O}_{Y,Q} \longrightarrow \mathcal{O}_{X,P}$, $f^*t \in \mathfrak{m}_P \subseteq \mathcal{O}_{X,P}$.

Write $v_P(t) = v_P(f^*t)$.

Def $f^*: \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(X)$, $[Q] \longmapsto \sum_{P \in f^{-1}(Q)} v_P(t)[P]$

Alternative Def

If $D \in \mathrm{Div}(Y)$, set $V = Y \setminus \mathrm{Supp}(D)$ and $s = 1 \in \Gamma(V, \mathcal{L}(D))$.

Note: $(s) = D$.

$f^*s \in \Gamma(f^{-1}(V), f^*\mathcal{L}(D))$. **Exercise** $f^*D = (f^*s) \in \mathrm{Div}(X)$.

Def R domain, M R-module. M is torsion free if
$\forall\, a\in R,\ m\in M:\quad am=0 \Rightarrow a=0$ or $m=0$.

Fact A f.g. torsion free module over a PID is free. (E.g. $R=\mathbb{Z}$.)

Def X non-sing. curve, $D=\sum n_i P_i \in \mathrm{Div}(X)$. Set $\deg(D)=\sum n_i \in \mathbb{Z}$.

WARNING: X not complete $\Rightarrow$ deg not def. on $Cl(X)$.

Prop $f:X\longrightarrow Y$ finite morphism of non-sing curves. $D\in \mathrm{Div}(X)$.
Then $\deg(f^*D)=\deg(f)\cdot\deg(D)$.

Proof Enough to show $\deg(f^*[Q])=\deg(f)$ for $Q\in Y$.
$V\subseteq Y$ open affine, $Q\in V$.
$f^{-1}(V)=\mathrm{Specm}(A)\subseteq X$, $A=\overline{k[V]}\subseteq k(X)$.
$Q\subseteq k[V]$ max. ideal. Set $B=A_Q=(k[V]-Q)^{-1}A$.
A f.g. $k[V]$-module $\Rightarrow$ B f.g. $k[V]_Q=\mathcal{O}_{Y,Q}$-module.
$\mathcal{O}_{Y,Q}$ DVR and B torsion free $\Rightarrow$ B free $\mathcal{O}_{Y,Q}$-module.
$\mathrm{rank}_{\mathcal{O}_{Y,Q}}(B)=\dim_{k(Y)} k(X)=\deg(f)$.
Let $m_Q=(t)\subseteq \mathcal{O}_{Y,Q}$. $\mathcal{O}_{Y,Q}/m_Q=k$.
$\therefore \dim_k(B/tB)=\deg(f)$.

Remark: Points in $f^{-1}(Q)$ $\leftrightarrow$ max ideals $P\subseteq A$ s.t. $P\cap k[V]=Q$
$\leftrightarrow$ max ideals in $A_Q=B$.

Write $f^{-1}(Q)=\{P_1,\dots,P_s\}$, $P_i\subseteq A$ max ideal.
$B=\bigcap_{i=1}^s B_{P_i}$ (normal Noeth domain) $\Rightarrow$ $tB=\bigcap_{i=1}^s (tB_{P_i}\cap B)$.

Claim (Chinese remainder thm) $B\xrightarrow{\cong}\bigoplus_{i=1}^s B/(tB_{P_i}\cap B)$ $\quad(*)$
Injective: Clear!
Surjective: $t\in P_i\ \forall i$. B_{P_i} DVR $\Rightarrow$ $tB_{P_i}=(P_iB_{P_i})^{n_i}$
$\Rightarrow tB_{P_i}\cap B\subseteq P_iB$ and $tB_{P_i}\cap B \not\subseteq P_jB$ for $j\neq i$.
$\therefore$ $(*)$ isomorphism after $\otimes B_{P_i}$, only i'th summand survives.

Now: $B/(tB_{P_i}\cap B)=\left(B/(tB_{P_i}\cap B)\right)_{P_i}=(B/tB)_{P_i}=B_{P_i}/tB_{P_i}=\mathcal{O}_{X,P_i}/t\mathcal{O}_{X,P_i}$
$\Rightarrow \dim_k(B/(tB_{P_i}\cap B))=v_{P_i}(t)$. $\quad\therefore \deg f^*[Q]=\sum \dim B/(tB_{P_i}\cap B)=\dim(B/tB)$
$=\deg(f)$.

$\square$

Lemma If $h \in k(Y)^*$ then $f^*((h)) = (f^*h)$,

$$f^*h = h \circ f \in k(X)^*.$$

Proof Let $P \in X$, $Q = f(P) \in Y$.

$m_P = (s) \subseteq \mathcal{O}_{X,P}$. $\quad m_Q = (t) \subseteq \mathcal{O}_{Y,Q}$.

$h = u t^m$, $\quad u \in \mathcal{O}_{Y,Q}$ unit.

$f^*t = v s^n$, $\quad v \in \mathcal{O}_{X,P}$ unit.

Coef of $[P]$ in $f^*((h)) = v_Q(h) \cdot v_P(t) = mn$.

$f^*h = f^*(ut^m) = f^*u \cdot (f^*t)^m = (f^*u) \cdot v^m \cdot s^{mn}$.

$\Rightarrow$ coef of $[P]$ in $(f^*h) = mn$.

$\square$

Hence group hom: $f^* : Cl(Y) \longrightarrow Cl(X)$.

Cor X complete non-sing curve, $f \in k(X)^* \Rightarrow \deg((f)) = 0$.

Proof f def. on $U \subseteq X$.

$f : U \longrightarrow \mathbb{A}^1 \subseteq \mathbb{P}^1$ extends to $f : X \longrightarrow \mathbb{P}^1$.

X complete $\Rightarrow$ $f : X \longrightarrow \mathbb{P}^1$ finite.

$k[\mathbb{A}^1] = k[t]$, $\quad f^*t = f \in k(X)$.

$(f) = (f^*t) = f^*((t)) \Rightarrow \deg((f)) = \deg(f^*((t)))$

$$= \deg(f) \cdot \deg((t))$$

$(t) = [0] - [\infty] \in \mathrm{Div}(\mathbb{P}^1) \Rightarrow \deg((t)) = 0$.

$\square$

$\therefore$ X complete non-sing curve $\Rightarrow$

$$\exists \ \deg : Cl(X) \longrightarrow \mathbb{Z}.$$

Notation: $D, D' \in Div(X)$. $D \sim D' \Leftrightarrow D = D' \in Cl(X)$.

Prop X complete non-sing curve.

X rational $\Leftrightarrow$ $\exists\, P \neq Q \in X$: $[P] \sim [Q]$.

Proof

$\Rightarrow$: $X \cong \mathbb{P}^1$. $P \sim Q$ $\forall\, P, Q \in \mathbb{P}^1$.

$\Leftarrow$: $\exists\, f \in k(X)^*$: $(f) = [P] - [Q] \in Div(X)$.

$f: X \longrightarrow \mathbb{P}^1$ morphism.

$(f) = (f^* t) = f^*((t)) = f^*([0] - [\infty])$

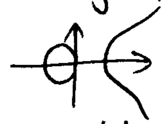
$\Rightarrow$ $f^*[0] = [P]$ $\Rightarrow$ $\deg(f) = 1$ $\Rightarrow$ f birat $\Rightarrow$ f iso.

$\square$

Alg. Geo. 12/4/2006 ①

Elliptic Curves.

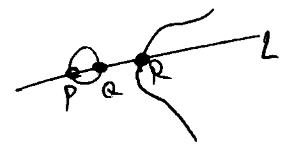
Def E an *elliptic curve* is a non-sing. plane curve $E \subseteq \mathbb{P}^2$ of deg. 3.

Example $V_+(zy^2 - x^3 + z^2x)$ 

Claim An elliptic curve can't be rational.

Exercise Set $\mathcal{O}_E(1) = \mathcal{O}_{\mathbb{P}^2}(1)|_E$. Then $\Gamma(E, \mathcal{O}_E(1)) = (S/I(E))_1$.
$S = k[x,y,z]$. $\therefore \dim_k \Gamma(E, \mathcal{O}_E(1)) = 3$.

Let $L \subseteq \mathbb{P}^2$ be a line.

$L.E = \sum_{P' \in L \cap E} I(L.E;P') \cdot P' = P + Q + R \in Div(E)$ 

Exercise $\mathcal{O}_E(1) = \mathcal{O}_{\mathbb{P}^2}([L])|_E = \mathcal{O}_E(L.E)$

$\therefore \exists\ D = L.E \in Div(E)$ s.t. $\deg(D) = 3$ and $\dim \Gamma(E, \mathcal{O}_E(D)) = 3$.

If $D \in Div(\mathbb{P}^1)$ has deg. 3 then $\mathcal{O}_{\mathbb{P}^1}(D) = \mathcal{O}_{\mathbb{P}^1}(3)$

$\Rightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D)) = k[x_0, x_1]_3 \Rightarrow \dim \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D)) = 4$.

Conclude: E is not rational.

Let $X \subseteq \mathbb{P}^2$ any non-sing curve.

Def $Cl^\circ(X) = \ker(\deg) \subseteq Cl(X)$. $\quad 0 \to Cl^\circ(X) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0$.

$Cl(X) = Cl^\circ(X) \oplus \mathbb{Z}$.

Let $L = V_+(f)$, $M = V_+(g) \subseteq \mathbb{P}^2$ be lines.

$X \cdot L = \sum I(X \cdot L; P) \cdot P = P_1 + \dots + P_n \in Div(X)$, $n = \deg(X)$.

$X \cdot M = Q_1 + \dots + Q_n$.

Exercise $f/g \in k(X)^*$. $(f/g) = X.L - X.M \in Div(X)$.

$\therefore X.L - X.M = 0 \in Cl^\circ(X)$.

$Cl^\circ(E) \oplus \mathbb{Z}$.

Thm Let $P_0 \in E$ be any point. Then $E \to Cl^\circ(E)$, $P \mapsto P - P_0$ is bijective. (So E is a group and $Cl(E) = E \oplus \mathbb{Z}$!)

Proof

Injective:

If $P - P_0 = Q - P_0 \in Cl^\circ(E)$ then $P \sim Q \Rightarrow P = Q$ (since E not rat.)

Surjective: Let $M \subseteq \mathbb{P}^2$ be tangent line to E at P_0.

$M.E = 2P_0 + R$, $R \in E$.

Let $D \in Cl^\circ(E)$. Write $D = \sum n_i (Q_i - P_0)$, $Q_i \in E$, $n_i \in \mathbb{Z}$.

Assume $n_i < 0$: $L =$ line through Q_i and R. ②

$L.E = Q_i + R + Q_i'$.

$L.E - M.E = Q_i + R + Q_i' - 2P_0 - R = 0 \in Cl^\circ(E)$

$\Rightarrow Q_i - P_0 = (-Q_i' - P_0)$

Replace Q_i with Q_i', and n_i with $-n_i$.

WLOG $n_i \geq 0 \quad \forall i$

Claim: $D = P - P_0 \in Cl^\circ(E)$ for some $P \in E$.

Induction over $\sum n_i$:

If $\sum n_i \geq 2$, let $(Q_1 - P_0)$ and $(Q_2 - P_0)$ have pos. coefs.

$L =$ line through Q_1, Q_2.

$L.E = Q_1 + Q_2 + Q' \in Div(E)$.

$L' =$ line through Q', P_0.

$L'.E = Q' + P_0 + Q''$.

$L.E - L'.E = Q_1 + Q_2 - P_0 - Q'' = 0 \in Cl^\circ(E)$

$\Rightarrow (Q_1 - P_0) + (Q_2 - P_0) = (Q'' - P_0) \in Cl^\circ(E)$.

Replace $(Q_1 - P_0) + (Q_2 - P_0)$ with $(Q'' - P_0)$.

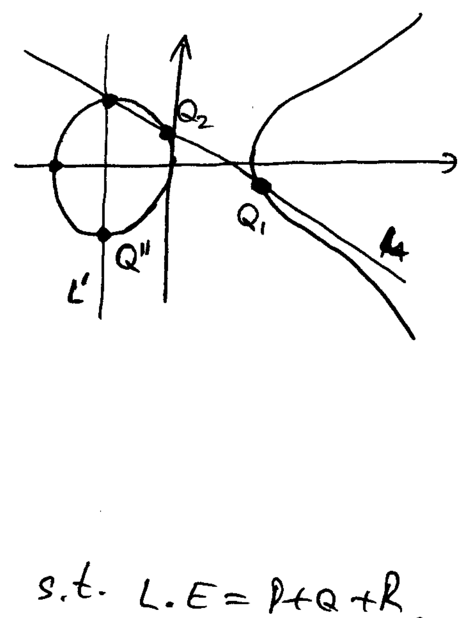
□

Example $char(k) \neq 2$. $\lambda \in k$, $\lambda \neq 0, 1$. $E_\lambda = V_+(zy^2 - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2$.

Take $P_0 = (0:1:0) \in E$ (point at ∞).

Let $\oplus$ be group op. on E.

I.e. $Q_1 \oplus Q_2 = Q'' \Leftrightarrow (Q_1 - P_0) + (Q_2 - P_0) = (Q'' - P_0)$.



Fact (later)

If E is any elliptic curve, $P_0 \in E$, $char(k) \neq 2$

then $\exists$ iso $E \xrightarrow{\cong} E_\lambda$, $P_0 \mapsto (0:1:0)$.

Thm E is an alg. group.

Pf $char(k) \neq 2$. WLOG $E = E_\lambda$, $P_0 = (0:1:0)$

Def $\varphi : E \times E \longrightarrow E$, $\varphi(P, Q) = R$ if $\exists$ line L s.t. $L.E = P + Q + R$.

Enough to show φ is a morphism. $(Q_1 \oplus Q_2 = \varphi(P_0, \varphi(Q_1, Q_2))$.)

Set $U_1 = E \cap D_+(z)$, $U_2 = E \cap D_+(y)$.

Then $E = U_1 \cap U_2$.

Enough to show $(U_i \times U_j) \cap \varphi^{-1}(U_\ell) \xrightarrow{\varphi} U_\ell$ morphism $\forall\, i,j,\ell \in \{0,1\}$.

$U_1 = V(y^2 - x(x-1)(x-\lambda)) \subseteq \mathbb{A}^2$.

$(U_1 \times U_1) \cap \varphi^{-1}(U_1) \xrightarrow{\varphi} U_1 \xrightarrow{x} k$ regular function given by

$$(x_1, y_1) \times (x_2, y_2) \longmapsto \begin{cases} \left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)^2 - (x_1 + x_2) + 1 + \lambda & \text{if } x_1 \neq x_2 \\ \left(\dfrac{x_1^2 + x_1x_2 + x_2^2 + \lambda - (1+\lambda)(x_1 + x_2)}{y_1 + y_2}\right)^2 + 1 + \lambda - (x_1 + x_2) & \text{if } y_1 + y_2 \neq 0. \end{cases}$$

__Exercise__

Do remaining 15 cases!

$\square$

__Differentials__

R ring, S commutative R-algebra. (have ring hom $R \to S$.)

M S-module. A fcn. $D: S \longrightarrow M$ is an __R-derivation__ if

(i) $D(fg) = f\,D(g) + g\,D(f) \quad \forall\, f, g \in S$.

(ii) $D(f+g) = D(f) + D(g)$

(iii) $D(f) = 0 \quad \forall\, f \in R$.

__Remark__ (iii) $\Leftrightarrow$ D is an R-homomorphism.

$\Rightarrow$: $f \in R \Rightarrow D(fg) = f\,D(g) + g\,D(f) = f\,D(g)$.

$\Leftarrow$: $D(1) = D(1 \cdot 1) = 1 \cdot D(1) + 1 \cdot D(1) = 2D(1) \Rightarrow D(1) = 0$.

D R-linear $\Rightarrow D(f) = D(f \cdot 1) = f \cdot D(1) = 0 \quad \forall\, f \in R$.

__Def__ F = free S-module with basis $\{d(f) \mid f \in S\} = \bigoplus_{f \in S} S \cdot d(f)$.

F' = submodule gen by all $d(f)$ for $f \in R$ and all $d(fg) - f\,d(g) - g\,d(f)$ for $f, g \in S$, and $d(f+g) - d(f) - d(g)$.

$\Omega_{S/R} := F/F'$. __Module of Kähler differentials__ of S over R.

$d = d_S = d_{S/R}: S \longrightarrow \Omega_{S/R}$, $f \mapsto d(f)$ __universal R-derivation__.

__Universal Property__

Given any R-deriv. $D: S \longrightarrow M$, $\exists!$ S-hom. $\tilde{D}: \Omega_{S/R} \longrightarrow M$ s.t. $D = \tilde{D} \circ d_{S/R}$.

__Exercise__ Let $P(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ and $f_1, \ldots, f_n \in S$, $D: S \longrightarrow M$ R-deriv.

Then $D(P(f_1, \ldots, f_n)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(f_1, \ldots, f_n) \cdot D(f_i)$.

__Consequence:__ If S gen. by $f_1, \ldots, f_n$ as <u>R-algebra</u>, then $\Omega_{S/R}$ gen by $d(f_1), \ldots, d(f_n)$ as S-module.

Prop Let $S = R[x_1, \dots, x_n]$. Then $\Omega_{S/R} = \bigoplus_{i=1}^{n} S \cdot d(x_i)$ free of rank n.

Proof Have surjection $S^n \longrightarrow \Omega_{S/R}$, $e_i \mapsto d(x_i)$.

Def. $D : S \longrightarrow S^n$, $p(x_1, \dots, x_n) \mapsto (\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n})$.

D is R-deriv. $\Rightarrow$ $\exists\ \tilde{D} : \Omega_{S/R} \longrightarrow S^n$, $d(x_i) \mapsto D(x_i) = e_i$. inverse map.

□

Prop Given ring homs. $R \to S \to T$, have exact seq. of T-modules

$$\Omega_{S/R} \otimes T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0$$

Proof $S \longrightarrow T \xrightarrow{d_T} \Omega_{T/R}$ is an R-derivation,

induces S-hom. $\varphi : \Omega_{S/R} \longrightarrow \Omega_{T/R}$, $\varphi(d_S(f)) = d_T(f)$.

Gives T-hom. $\Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R}$, $\omega \otimes h \mapsto h \cdot \varphi(\omega)$.

Note: $\text{Image}(\Omega_{S/R} \otimes T \longrightarrow \Omega_{T/R})$ = submod. gen. by $d_T(f)$, $f \in S$.

$\Rightarrow$ $\Omega_{T/R} / \text{Image} = \Omega_{T/S}$.

□

Note: $I \le S$ ideal. $T = S/I$. Then I/I^2 is a T-module.

$T \times I/I^2 \longrightarrow I/I^2$, $(f+I) \cdot (h + I^2) = fh + I^2$.

Prop $T = S/I$. Have exact seq. of T-modules

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow 0.$$

$h + I^2 \mapsto d_S(h) \otimes 1$.

Proof Set M = image of I/I^2 in $\Omega_{S/R} \otimes_S T$.

i.e. M gen. by $\{ d_S(h) \otimes 1 \mid h \in I \}$.

Def. $D : T \longrightarrow (\Omega_{S/R} \otimes_S T)/M$, $D(f+I) = (d_S(f) \otimes 1) + M$.

This is an R-deriv. $\Rightarrow$

$\exists\ \tilde{D} : \Omega_{T/R} \longrightarrow (\Omega_{S/R} \otimes_S T)/M$ T-hom.

$d_T(f+I) \mapsto (d_S(f) \otimes 1) + M$.

□

S commutative R-algebra.

$$\Omega_{S/R} = \Omega_S := \Big(\bigoplus_{f\in S} S\cdot d(f)\Big) \Big/ \left\langle \begin{matrix} d(fg) - f\,d(g) - g\,d(f) \\ d(f+g) - d(f) - d(g) \\ d(r) \end{matrix} \quad \begin{matrix} f,g\in S \\ r\in R \end{matrix} \right\rangle$$

$d = d_S = d_{S/R} : S \longrightarrow \Omega_{S/R}$ univ. R-derivation.

Prop $S = R[x_1,\dots,x_n] \Rightarrow \Omega_{S/R} = S\,d(x_1) \oplus \dots \oplus S\,d(x_n)$. $d(f) = \sum \frac{\partial f}{\partial x_i} d(x_i)$

Prop Given ring homs. $R \to S \to T$, have exact seq. of T-modules

$$\Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0$$

Proof $S \longrightarrow T \xrightarrow{d_T} \Omega_{T/R}$ is an R-derivation $\Rightarrow$

induces S-hom. $\varphi : \Omega_{S/R} \longrightarrow \Omega_{T/R}$, $\varphi(d_S(f)) = d_T(f)$.

Gives T-hom: $\tilde\varphi : \Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R}$, $\omega \otimes h \longmapsto h\cdot\varphi(\omega)$

Note: $\mathrm{Image}(\tilde\varphi)$ = submodule gen. by $d_T(f)$, $f\in S$.

$\Rightarrow \Omega_{T/R} / \mathrm{Im}(\tilde\varphi) = \Omega_{T/S}$.

□

Note: $I \subseteq S$ ideal, $T = S/I$. Then I/I^2 is a T-module.

$$T \times I/I^2 \longrightarrow I/I^2 , \quad (f+I)\cdot(h+I^2) = fh + I^2.$$

Prop $T = S/I$. Have exact seq. of T-modules

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow 0$$
$$h + I^2 \longmapsto d_S(h) \otimes 1.$$

Proof Set M = image of I/I^2 in $\Omega_{S/R} \otimes T$.

Then M gen. by $\{ d_S(h) \otimes 1 \mid h \in I \}$.

Def. $D : T \longrightarrow (\Omega_{S/R} \otimes T)/M$, $D(f+I) = (d_S(f) \otimes 1) + M$

This is an R-derivation $\Rightarrow$

$\exists!$ T-hom. $\tilde D : \Omega_{T/R} \longrightarrow (\Omega_{S/R} \otimes T)/M$, $d_T(f+I) \longmapsto (d_S(f)\otimes 1) + M$.

□

Example $S = R[x_1,\dots,x_n]$, $I = (f_1,\dots,f_p) \subseteq S$. $T = S/I$.

Then $\Omega_{S/R} \otimes_S T = \bigoplus_{i=1}^n T\cdot dx_i = T^n$.

Image of f_i under $I/I^2 \longrightarrow T^n$ is $d_S(f_i) \otimes 1 = \left(\overline{\frac{\partial f_i}{\partial x_1}}, \dots, \overline{\frac{\partial f_i}{\partial x_n}}\right) \in T^n$.

Set $J = \left(\frac{\partial f_i}{\partial x_j}\right) \in \mathrm{Mat}_{p\times n}(T)$. Jacobi-matrix.

Image of f_i = $e_i \cdot J$.

$$\Rightarrow \Omega_{T/R} = \mathrm{coker}(I/I^2 \longrightarrow \Omega_{S/R}\otimes T) = \mathrm{coker}(T^p \xrightarrow{\cdot J} T^n).$$

Eg. $T = k[x,y]/(y^2 - x^3 + x)$ $J = (1 - 3x^2,\ 2y)$.

$$\Omega_{T/k} = T \oplus T / \langle (1-3x^2)e_1 + 2y\, e_2 \rangle$$

<u>Prop</u> S an R-algebra. $U \subseteq S$ mult. subset. Then $\Omega_{U^{-1}S/R} = U^{-1}\Omega_{S/R}$.

<u>Proof</u>

$S \longrightarrow U^{-1}S \longrightarrow \Omega_{U^{-1}S/R}$ R-deriv., induces

$\Omega_{S/R} \longrightarrow \Omega_{U^{-1}S/R}$ S-hom. $d_S(f) \longmapsto d(f)$.

Induces $U^{-1}\Omega_{S/R} \longrightarrow \Omega_{U^{-1}S/R}$, $d_S(f)/u \longmapsto u^{-1}d(f)$.

<u>Def</u> $D : U^{-1}S \longrightarrow U^{-1}\Omega_{S/R}$, $D(s/u) = (u\,d(s) - s\,d(u))/u^2$.

<u>Exercise</u>: D is well defined R-derivation!

Induces $\tilde{D} : \Omega_{U^{-1}S/R} \longrightarrow U^{-1}\Omega_{S/R}$ inverse map.

□

X top. space, $\mathcal{R}$, $\mathcal{S}$ sheaves of rings on X, $\mathcal{R} \longrightarrow \mathcal{S}$ ring. hom.

<u>Def</u> $\text{pre-}\Omega_{\mathcal{S}/\mathcal{R}}(U) = \Omega_{\mathcal{S}(U)/\mathcal{R}(U)}$

$V \subseteq U$: $\mathcal{S}(U) \longrightarrow \mathcal{S}(V) \longrightarrow \text{pre-}\Omega(V)$ is an $\mathcal{R}(U)$-deriv.

$\Rightarrow$ induces $\text{pre-}\Omega(U) \longrightarrow \text{pre-}\Omega(V)$ $\mathcal{S}(U)$-hom.

$\Omega_{\mathcal{S}/\mathcal{R}} := (\text{pre-}\Omega_{\mathcal{S}/\mathcal{R}})^+$.

Let $\varphi : X \longrightarrow Y$ morphisms of varieties. Gives ring hom. $\varphi^* : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X.

<u>Def</u> $\Omega_{X/Y} = \Omega_{\mathcal{O}_X/\varphi^{-1}\mathcal{O}_Y}$ <u>relative cotangent sheaf</u>.

Special case: $X \longrightarrow \{\text{point}\}$. $\Omega_X = \Omega_{X/k} := \Omega_{X/\{pt\}}$. <u>Cotangent sheaf</u>.

<u>Lemma</u> $\varphi : X \to Y$ morphism of affine varieties. Then $\text{pre-}\Omega_{X/Y}(X) = \Omega_{k[X]/k[Y]}$

<u>Proof</u> Set $S = \Gamma(X, \varphi^{-1}\mathcal{O}_Y)$. $k[Y] \to S \to k[X]$ ring hom.

$\Rightarrow \Omega_{S/k[Y]} \otimes_S k[X] \longrightarrow \Omega_{k[X]/k[Y]} \longrightarrow \Omega_{k[X]/S} \to 0$ exact.

$\Omega_{k[X]/S} = \text{pre-}\Omega(X)$

Enough: $\Omega_{S/k[Y]} \otimes_S k[X] = 0$.

Let $f \in S = \varinjlim_{V \supseteq \varphi(X)} \mathcal{O}_Y(V)$. Then $f \in \mathcal{O}_Y(V)$, $V \supseteq \varphi(X)$.

$V = \bigcup_{i=1}^{n} D(h_i)$, $h_i \in k[Y]$.

$f \in k[D(h_i)] = k[Y]_{h_i} \Rightarrow h_i^N f \in k[Y] \quad \forall i$, some $N \gg 0$.

$\Rightarrow h_i^N d_S(f) = d_S(h_i^N f) = 0 \in \Omega_{S/k[Y]}$.

Now $X = \bigcup_{i=1}^{n} D(h_i^N) \Rightarrow (h_1^N, \dots, h_n^N) = (1) \subseteq k[X]$

$h_i^N \cdot (d_S(f) \otimes 1) = 0 \Rightarrow d_S(f) \otimes 1 = 0 \in \Omega_{S/k[X]} \otimes_S k[X]$.

□

Prop $\varphi : X \to Y$ morphism of affine varieties. Then $\Omega_{X/Y} = \widetilde{\Omega_{k[X]/k[Y]}}$.

Proof: next time.

Cor $\Omega_{X/Y}$ is always coherent.

Lemma (A, m) local Noeth. domain. N f.g. A-module.
Set $r = \dim_{A/m}(N/mN)$. If $r \leq \dim_{A_0}(N_0)$ then $N \cong A^r$.

Proof
NAkayama's lemma $\Rightarrow$ N gen. by r elements.
$\Rightarrow \exists$ exact seq. $\quad 0 \to K \to A^r \to N \to 0$.
Localize: $\quad 0 \to K_0 \to A_0^r \to N_0 \to 0$.
Last two v.s. have same dim $\Rightarrow K_0 = 0$
But $K \subseteq A^r$ is torsion free $\Rightarrow K = 0$.
□

Recall $X \subseteq \mathbb{A}^n$ irred. closed. $I = I(X) = (f_1, \dots, f_s)$. $p \in X$.
Set $M = I(\{p\}) \subseteq k[x_1, \dots, x_n]$.
$M/M^2 \xrightarrow{\sim} k^n$. $\quad h + M^2 \mapsto \left(\frac{\partial h}{\partial x_1}(p), \dots, \frac{\partial h}{\partial x_n}(p)\right)$.
Exact: $\quad I + M^2/M^2 \longrightarrow M/M^2 \longrightarrow m_p/m_p^2 \longrightarrow 0$.
$\Rightarrow k^s \xrightarrow{\cdot J(p)} k^n \xrightarrow{\phi} m_p/m_p^2 \longrightarrow 0$, $\quad J = \left(\frac{\partial f_i}{\partial x_j}\right)$ Jacobi.
If $h \in M$ then $\phi\left(\frac{\partial h}{\partial x_1}(p), \dots, \frac{\partial h}{\partial x_n}(p)\right) = h + m_p^2$.
Note: $\operatorname{rank}\left(k(X)^s \xrightarrow{\cdot J} k(X)^n\right) \leq c = \operatorname{codim}(X; \mathbb{A}^n)$.
(If h is a $(c+1)$-minor of J, then $h \in k[X]$.
$h(p) = (c+1)$-minor of $J(p) \Rightarrow h(p) = 0 \ \forall p \in X \Rightarrow h = 0 \in k[X]$.)

Thm Assume X irred. variety of dim. r, $p \in X$.

p non-sing. point $\iff$ $\Omega_{X,p} \cong \mathcal{O}_{X,p}^{\oplus r}$.

If $\mathfrak{m}_p$ gen. by $h_1, \dots, h_r \in \mathfrak{m}_p$ then $dh_1, \dots, dh_r \in \Omega_{X,p}$ is a basis.

Proof WLOG: $X \subseteq \mathbb{A}^n$ affine. $I(X) = (f_1, \dots, f_s)$. $J = \left(\frac{\partial f_i}{\partial x_j}\right)$.

$$k[X]^s \xrightarrow{\cdot J} k[X]^n \longrightarrow \Omega_{k[X]/k} \longrightarrow 0.$$

$$\Rightarrow \mathcal{O}_{X,p}^{\oplus s} \xrightarrow{\cdot J} \mathcal{O}_{X,p}^{\oplus n} \longrightarrow \Omega_{X,p} \qquad (*).$$

$$/\mathfrak{m}_p : \quad k^s \xrightarrow{\cdot J(p)} k^n \longrightarrow \Omega_{X,p}/\mathfrak{m}_p\Omega_{X,p} \longrightarrow 0.$$

$$\therefore \quad \mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{\sim} \Omega_{X,p}/\mathfrak{m}_p\Omega_{X,p} \qquad (**)$$

$$h + \mathfrak{m}_p^2 \longmapsto \sum_{j=1}^n \frac{\partial h}{\partial x_j}(p)\, dx_j \quad . \qquad (\text{apply } \phi^{-1})$$

$\Omega_{X,p}$ free of rank $r \Rightarrow \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = r \Rightarrow p$ non-sing.

Suppose $\dim(\mathfrak{m}_p/\mathfrak{m}_p^2) = r$.

$(*) \Rightarrow k(X)^s \xrightarrow{\cdot J} k(X)^n \longrightarrow (\Omega_{X,p})_0 \longrightarrow 0$ exact.

Note $\Rightarrow$ $r = \dim_k(\Omega_{X,p}/\mathfrak{m}_p\Omega_{X,p}) \leq \dim_{k(X)}(\Omega_{X,p})_0$

So Lemma $\Rightarrow$ $\Omega_{X,p} \cong \mathcal{O}_{X,p}^{\oplus r}$.

Finally, if $h_1, \dots, h_r$ generate $\mathfrak{m}_p$ then $(**)$ shows that $\overline{dh_1}, \dots, \overline{dh_r}$ is basis for $\Omega_{X,p}/\mathfrak{m}_p\Omega_{X,p}$.

$\Rightarrow_{NAK}$ $dh_1, \dots, dh_r$ basis for $\Omega_{X,p}$.

$\square$

$\varphi : X \to Y$ morphism of varieties. $\varphi^* : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ ring. hom.

<u>Def</u> $\Omega_{X/Y} = \Omega_{\mathcal{O}_X/\varphi^{-1}\mathcal{O}_Y} = \left(U \mapsto \Omega_{\mathcal{O}_X(U)/\varphi^{-1}\mathcal{O}_Y(U)} \right)^+ = (\text{pre-}\Omega_{X/Y})^+$.

$\Omega_X = \Omega_{X/\{\text{point}\}}$.

<u>Lemma</u> $\varphi : X \to Y$ morphism of affine varieties. Then $\text{pre-}\Omega_{X/Y}(X) = \Omega_{k[X]/k[Y]}$

<u>Proof</u> Set $S = \Gamma(X, \varphi^{-1}\mathcal{O}_Y)$. $k[Y] \to S \to k[X]$ ring. homs.

$\Rightarrow \Omega_{S/k[Y]} \otimes_S k[X] \xrightarrow[?]{0} \Omega_{k[X]/k[Y]} \to \underset{\text{pre-}\Omega(X)}{\underset{\parallel}{\Omega_{k[X]/S}}} \to 0$ exact.

Enough: First map is zero.

Let $f \in \text{Image}(S \to k[X])$. Must show $df = 0 \in \Omega_{k[X]/k[Y]}$.

$\exists$ open cover $X = \bigcup_{i=1}^n U_i$ s.t. $f|_{U_i} \in$ image of $\Gamma(U_i, \text{pre-}\varphi^{-1}\mathcal{O}_Y) = \varinjlim_{V \supseteq \varphi(U_i)} \mathcal{O}_Y(V)$

WLOG $U_i = X_{g_i}$, $g_i \in k[X]$.

Enough to show $df = 0 \in \left(\Omega_{k[X]/k[Y]}\right)_{g_i}$

(since then $g_i^N \cdot df = 0 \ \forall i$ and $(g_1^N, \dots, g_n^N) = (1) = k[X]$.)

But $\left(\Omega_{k[X]/k[Y]}\right)_{g_i} = \Omega_{k[U_i]/k[Y]}$

Replace X with U_i; ~~may assume $f \in$ image of $\Gamma(X, \dots)$~~

May assume $f \in$ image of $\Gamma(X, \text{pre-}\varphi^{-1}\mathcal{O}_Y) = \varinjlim_{V \supseteq \varphi(X)} \mathcal{O}_Y(V)$.

I.e. $\exists V \subseteq Y$ open, $f' \in \mathcal{O}_Y(V)$ s.t. $\varphi(X) \subseteq V$ and $f = \varphi^*(f') \in k[X]$.

$V = \bigcup_{i=1}^m Y_{h_i}$, $h_i \in k[Y]$.

$f' \in k[Y]_{h_i} \Rightarrow h_i^N \cdot f' \in k[Y] \ \forall i$, some $N \gg 0$.

$\Rightarrow h_i^N \cdot df = d(h_i^N \cdot f) = 0 \in \Omega_{k[X]/k[Y]}$.

$X = \bigcup_{i=1}^m X_{h_i} \Rightarrow (h_1^N, \dots, h_m^N) = (1) = k[X] \Rightarrow df = 0 \in \Omega_{k[X]/k[Y]}$.

$\square$

<u>Prop</u> $\varphi : X \to Y$ morphism of affine varieties. Then $\Omega_{X/Y} = \widetilde{\Omega_{k[X]/k[Y]}}$.

<u>Proof</u> Set $\Omega = \Omega_{k[X]/k[Y]}$. The map $\Omega \xrightarrow{\sim} \Gamma(X, \text{pre-}\Omega_{X/Y}) \to \Gamma(X, \Omega_{X/Y})$

induces $\mathcal{O}_X$-hom. $\tilde{\Omega} \to \Omega_{X/Y}$.

Have global morphism, enough to show ~~stalks are the same~~ iso. on stalks.

Let $f \in k[X]$. $\Gamma(D(f), \tilde{\Omega}) = \Omega_f = \Omega_{k[X_f]/k[Y]} = \Gamma(X_f, \text{pre-}\Omega_{X/Y})$

$\Rightarrow$ same stalks.

$\square$

Cor $\Omega_{X/Y}$ is coherent.

Proof Given morphism $\varphi: X \to Y$, take open affine cover $Y = \bigcup V_i$ and open affine covers $\varphi^{-1}(V_i) = \bigcup U_{ij} \subseteq X$.
Then $X = \bigcup U_{ij}$ and
$$\Omega_{X/Y}|_{U_{ij}} \cong \widetilde{\Omega_{k[U_{ij}]/k[V_i]}}.$$
$\square$

Thm X irred. variety of dim. r. $P \in X$. P non-sing $\Leftrightarrow \Omega_{X,P} \cong \mathcal{O}_{X,P}^{\oplus r}$.
If $m_P \subseteq \mathcal{O}_{X,P}$ gen by $h_1, \dots, h_r$ then $dh_1, \dots, dh_r \in \Omega_{X,P}$ is a basis.

Cor X irred. variety. X non-singular $\Leftrightarrow \Omega_{X/k} = \Omega_X$ locally free $\mathcal{O}_X$-module.

Example $X = \mathbb{P}^1$. $\Omega_{\mathbb{P}^1}$ line bundle. Proj. coord ring $k[x_0, x_1]$.
Set $t = \frac{x_1}{x_0} \in k(\mathbb{P}^1)$. Then $dt \in \Gamma(U_0, \Omega_{\mathbb{P}^1})$, $U_0 = D_+(x_0) = \mathbb{A}^1$.
Find $(dt) \in \mathrm{Div}(\mathbb{P}^1)$.
If $p \in U_0$ then $t-p$ generate $m_p \Rightarrow \Omega_{\mathbb{P}^1,p}$ gen. by $d(t-p) = dt$.
$\Rightarrow v_p(dt) = 0$.
$U_1 = D_+(x_1)$, $k[U_1] = k[s]$, $s = x_0/x_1$.
$dt = d(s^{-1}) = -s^{-2}\,ds \Rightarrow v_\infty(dt) = v_\infty(s^{-2}) = -2$.
$\therefore (dt) = -2[\infty] \Rightarrow \Omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

Exercise E elliptic curve $\Rightarrow \Omega_E = \mathcal{O}_E$. $\therefore$ E still not rational.

Linear systems

X non-sing proj. variety.
$D = \sum n_Y [Y] \in \mathrm{Div}(X)$ is effective ($D \geq 0$) if $n_Y \geq 0 \ \forall Y$.
Given any $D \in \mathrm{Div}(X)$, set $|D| = \{D' \in \mathrm{Div}(X) \mid D' \sim D \text{ and } D' \geq 0\}$ complete linear system

Thm X projective and $\mathcal{F}$ coherent $\mathcal{O}_X$-module $\Rightarrow \dim_k \Gamma(X, \mathcal{F}) < \infty$.

Def $\ell(D) = \dim_k \Gamma(X, \mathcal{O}_X(D))$.

Thm $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D))) \xrightarrow{\sim} |D|$, $s \mapsto (s)$ is bijective.

Proof If $s \in \Gamma(X, \mathcal{O}_X(D))$ global section then $(s) \geq 0$, so map well def.
Surjective: $D' \in |D|$. Then $D' \sim D \Rightarrow D' = D + (f)$, $f \in k(X)^*$.
$s := f \in \Gamma(X, \mathcal{O}_X(D))$ since $v_Y(f) \geq -n_Y \ \forall Y$.
If $s_0 := 1 \in \Gamma(U, \mathcal{O}_X(D))$ then $(s) = (fs_0) = (f) + (s_0) = (f) + D = D'$.

<u>Injective</u>: Let $s_1, s_2 \in \Gamma(X, \mathcal{O}_X(D))$. Assume $(s_1) = (s_2) \in \mathrm{Div}(X)$.

Then $(s_1/s_2) = (s_1) - (s_2) = 0 \Rightarrow s_1/s_2 \in k[X] = k$.

□

<u>Lemma</u> X non-sing. curve, $D \in \mathrm{Div}(X)$.

If $\ell(D) \neq 0$ then $\deg(D) \geq 0$. If $\ell(D) \neq 0$ and $\deg(D) = 0$ then $D \sim 0$.

<u>Proof</u>

$\ell(D) \neq 0 \Rightarrow |D| \neq \emptyset \Longrightarrow D \sim D' \geq 0. \Rightarrow \deg(D) = \deg(D') \geq 0$.

If $\deg(D) = 0$ then $D' = 0$.

□

<u>Riemann-Roch Theorem</u>

X non-sing. projective curve.

~~Def Genus~~ $\cancel{g = \dim_k \Gamma(X, \Omega_X)}$

<u>Def</u>. $K \in \mathrm{Div}(X)$ is a canonical divisor if $\mathcal{O}_X(K) \cong \Omega_X$.

<u>Def</u> Genus of X: $g = \ell(K) = \dim_k \Gamma(X, \Omega_X)$.

<u>Examples</u> • $\Omega_{\mathbb{P}^1} = \mathcal{O}(-2) \Rightarrow g = 0$

• E elliptic curve. $\Omega_E \cong \mathcal{O}_E \Rightarrow g = 1$

<u>R-R. Thm</u>

$$\boxed{\ell(D) - \ell(K-D) = \deg(D) + 1 - g \quad \forall D \in \mathrm{Div}(X).}$$

<u>Example</u> $X = \mathbb{P}^1$. $K = -2[P]$. $\ell(nP) - \ell(-2P-nP) = n+1-0$.

$n \geq 0$: $(n+1) - 0 = n+1-0$ ✓

$n = -1$: $0 - 0 = -1+1-0$ ✓

$n \leq -2$: $0 - (-n-1) = n+1-0$ ✓

<u>Example</u> Take $D = K$. Note $\ell(K) = g$.

$\ell(K) - \ell(K-K) = \deg(K) + 1 - g$

$g - 1 = \deg(K) + 1 - g \Rightarrow \deg(K) = 2g-2$.

<u>Cor</u> A non-sing. curve is either affine or projective.

<u>Proof</u> $X = C_K - \{P_1, \dots, P_n\}$, $K = k(X)$.

If $m \gg 0$ then $\ell(mP_i) = m+1-g \geq 1$.

$\Rightarrow \exists f_i \in k(X)^*$ s.t. $(f_i) = -m[P_i]$ + effective divisor.

Set $f = \sum f_i \in k(X)^*$.

Then f is defined exactly on $X \subseteq C_K \Rightarrow X = \mathrm{Specm}(\overline{k[f]})$.

□

Cor X rational $\Leftrightarrow$ $g=0$.

Proof $\Rightarrow$: already done.

Suppose $g=0$. Let $P \neq Q \in X$. $D = P - Q \in \mathrm{Div}(X)$.

$RR \Rightarrow \ell(D) \geq \deg(D)+1-g = 0+1-0 = 1$.

Lemma $\Rightarrow D \sim 0 \Rightarrow P \sim Q \Rightarrow X \cong \mathbb{P}^1$.

$\square$

Cor X non-sing. curve of genus $g=1$, $P_0 \in X$ any point. $\mathrm{char}(k) \neq 2$.

Then $\exists$ iso. $X \xrightarrow{\sim} E_\lambda = V_+(zy^2 - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2$, $\lambda \neq 0,1$.

$P_0 \longmapsto (0:1:0)$.

Proof

$\deg(K) = 2g-2 = 0$.

$RR \Rightarrow \ell(nP_0) = n+1-1 = n \quad \forall n \geq 1$.

$k = \Gamma(\mathcal{O}_X) = \Gamma(\mathcal{O}_X(P_0)) \subsetneq \Gamma(\mathcal{O}_X(2P_0)) \subsetneq \Gamma(\mathcal{O}_X(3P_0)) \subsetneq \cdots \subseteq k(X)$.

Take $x \in \Gamma(X, \mathcal{O}_X(2P_0)) \setminus k$.

$v_{P_0}(x) = -2$. $(x) = A + B - 2P_0$, $A, B \in X$.

$x : X \longrightarrow \mathbb{P}^1$ morphism. $x^*([0] - [\infty]) = A + B - 2P_0$.

$\Rightarrow x^*([0]) = A+B \Rightarrow [k(X) : k(x)] = \deg(x) = \deg(x^*([0])) = 2$.

Take $y \in \Gamma(X, \mathcal{O}_X(3P_0)) \setminus \Gamma(X, \mathcal{O}_X(2P_0))$.

Then $y \notin k(x)$ since $v_{P_0}(y) = -3$ is odd.

$\therefore k(X) = k(x,y)$.

$\{1, x, y, x^2, xy\}$ basis for $\Gamma(X, \mathcal{O}_X(5P_0))$.

$1, x, y, x^2, xy, x^3, y^2 \in \Gamma(\mathcal{O}_X(6P_0))$; $\dim = 6$.

$\Rightarrow \exists$ relation s.t. coefs of x^3 and y^2 are $\neq 0$.

Replace x, y with scalar multiples:

$$y^2 + b_1xy + b_0y = x^3 + a_2x^2 + a_1x + a_0 \quad ; \quad a_i, b_i \in k.$$

Replace y with $y + \frac{1}{2}(b_1x + b_0)$:

$$y^2 = (x-a)(x-b)(x-c) \quad , \quad a, b, c \in k.$$

Claim: $a \neq b \neq c \neq a$. ⑤

$\varphi : X \setminus \{P_0\} \xrightarrow{\sim} C = V(y^2 - (x-a)(x-b)(x-c)) \subseteq \mathbb{A}^2$ birational since $k(X) = k(x,y)$

$P \longmapsto (x(P), y(P))$.

$a=b=c$: $C = \prec$ rational $\Rightarrow X$ rational ↯.

$a=b\neq c$: $C = \alpha$ rational ↯.

$\therefore$ Claim true.

Replace x with $\frac{x-a}{b-a}$; rescale y:

$$y^2 = x(x-1)(x-\lambda), \qquad \lambda \neq 0,1 \qquad \lambda = \frac{c-a}{b-a}.$$

Finally, birat map $X - \{P_0\} \longrightarrow C$ extends to iso

$X \xrightarrow{\sim} E_\lambda$, $P_0 \mapsto (0:1:0)$.

$\square$

X complete non-sing curve.

$K \in \mathrm{Div}(X)$ canonical divisor if $\Omega_X \cong \mathcal{O}_X(K)$.

Genus of X: $g = \ell(K) = \dim_k \Gamma(X, \Omega_X)$.

Riemann-Roch: $\ell(D) + \ell(K-D) = \deg(D) + 1 - g \quad \forall D \in \mathrm{Div}(X)$.

Set $D = K \Rightarrow \deg(K) = 2g-2$.

Cor A non-sing. curve is either affine or projective.

Proof C non-sing curve $\Rightarrow C = C_K \setminus \{P_1, \dots, P_n\}$, $K = k(C)$.

If $m \gg 0$ then $\ell(m P_i) = m+1-g \geq 2$

$\Rightarrow \exists f_i \in k(C_K)^*$ s.t. $(f_i) = -r_i[P_i] +$ effective divisor $\in \mathrm{Div}(C_K)$

Set $f = \sum f_i \in k(C_K)^*$.

Then f is def. exactly on $C \subseteq C_K \Rightarrow C = \mathrm{Specm}(\overline{k[f]})$.

□

Cor X rational $\Leftrightarrow g = 0$.

Proof: $\Rightarrow$: $\mathrm{genus}(\mathbb{P}^1) = 0$.

$\Leftarrow$: Let $P \neq Q \in X$. $D = P - Q \in \mathrm{Div}(X)$.

RR $\Rightarrow \ell(D) \geq \deg(D) + 1 - g = 1$

$\Rightarrow |D| \neq \emptyset$.

$\exists D' \geq 0$ s.t. $D' \sim D$. $\deg(D') = 0 \Rightarrow D' = 0$.

$\therefore P \sim Q \Rightarrow X \cong \mathbb{P}^1$.

□

Cor X complete non-sing curve of genus $g=1$, $P_0 \in X$ any point. $\mathrm{char}(k) \neq 2$.

Then $\exists$ iso. $X \xrightarrow{\cong} E_\lambda = V_+(zy^2 - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2$, $P_0 \mapsto (0:1:0)$. $\lambda \neq 0, 1$.

Proof

$\deg(K) = 2g - 2 = 0$.

RR $\Rightarrow \ell(nP_0) = n+1-1 = n \quad \forall n \geq 1$.

$k = \Gamma(\mathcal{O}_X) = \Gamma(\mathcal{O}_X(P_0)) \subsetneq \Gamma(\mathcal{O}_X(2P_0)) \subsetneq \cdots \subseteq k(X)$.

Take $x \in \Gamma(\mathcal{O}_X(2P_0)) \setminus k$.

$v_{P_0}(x) = -2$. $(x) = A + B - 2P_0$, $A, B \in X$.

$x: X \longrightarrow \mathbb{P}^1$ morphism. $x^*([0] - [\infty]) = A + B - 2P_0$

$\Rightarrow x^*([0]) = A + B \Rightarrow [k(X) : k(x)] = \deg(x) = \deg(x^*([0])) = 2$.

Take $y \in \Gamma(\mathcal{O}_X(3P_0)) \setminus \Gamma(\mathcal{O}_X(2P_0))$.

Then $y \notin k(x)$ since $v_{P_0}(y) = -3$ is odd!
$\therefore\ k(X) = k(x,y)$.
$\{1, x, y, x^2, xy\}$ basis for $\Gamma(\mathcal{O}_X(5P_0))$.
$1, x, y, x^2, xy, x^3, y^2 \in \Gamma(\mathcal{O}_X(6P_0))$; $\dim = 6$.
$\Rightarrow \exists$ linear rel. s.t. coefs of $x^3, y^2 \neq 0$.
Replace x, y with scalar multiples:

$$y^2 + b_1 xy + b_0 y = x^3 + a_2 x^2 + a_1 x + a_0 \quad ; \quad a_i, b_i \in k.$$

Replace y with $y + \frac{1}{2}(b_1 x + b_0)$:

$$y^2 = (x-a)(x-b)(x-c) \quad ; \quad a, b, c \in k.$$

<u>Claim</u>: $a \neq b \neq c \neq a$:

$$\varphi : X \setminus \{P_0\} \xrightarrow{\sim} C = V(y^2 - (x-a)(x-b)(x-c)) \subseteq \mathbb{A}^2$$
$$P \longmapsto (x(P), y(P))$$

birational since $k(X) = k(x,y)$.

$a = b = c$: $C =$ rational $\Rightarrow X$ rational $\lightning$.
$a = b \neq c$: $C =$ rational $\lightning$.

Replace x with $\frac{x-a}{b-a}$, rescale y:

$$y^2 = x(x-1)(x-\lambda) \quad , \quad \lambda = \frac{c-a}{b-a} \neq 0, 1$$

Finally, $X - \{P_0\} \longrightarrow C$ extends to iso. $X \xrightarrow{\cong} E_\lambda$, $P_0 \mapsto (0:1:0)$.
□

<u>Lemma</u> X complete non-sing curve of genus g. Let $P_0, Q_0, Q_1, \dots, Q_g \in X$.
Then $\exists\ P_1, \dots, P_g \in X$ s.t. $\sum_{i=0}^{g} P_i \sim \sum_{i=0}^{g} Q_i$.
<u>Proof</u>
WLOG $Q_i \neq P_0\ \forall i$. Set $D = \sum Q_i$ $\quad \ell(D) \geq \deg(D) + 1 - g \geq 2$.
$\Rightarrow \exists\ h \in \Gamma(X, \mathcal{O}_X(D))$ s.t. $h \notin k$.
Set $f = h - h(P_0)$.
Then $(f) = -D + P_0 + P_1 + \dots + P_g$.
□

<u>Cor</u> The map $X^g = \underbrace{X \times \dots \times X}_{g} \longrightarrow Cl^0(X)$, $(P_1, \dots, P_g) \mapsto \sum P_i - gP_0$ is surjective.
<u>Proof</u>
<u>Note</u>: Let $Q \in X$. Lemma $\Rightarrow \exists\ P_1, \dots, P_g \in X$ s.t. $(g+1)P_0 \sim Q + P_1 + \dots + P_g$.
$\Rightarrow -(Q - P_0) = \sum_{i=1}^{g} (P_i - P_0) \in Cl^0(X)$.
Let $D = \sum n_Q (Q - P_0) \in Cl^0(X)$.
Note $\Rightarrow$ may assume $n_Q \geq 0\ \forall Q$. Lemma $\Rightarrow$ may assume $\sum n_Q \leq g$. □

Blow-up of varieties.

Y affine variety, $X \subseteq Y$ closed subvariety.

$I = I(X) = (f_0, \dots, f_n) \subseteq k[Y]$.

$\varphi : Y \setminus X \longrightarrow \mathbb{P}^n$, $\quad \varphi(y) = (f_0(y) : \cdots : f_n(y))$

Def $\quad Bl_X(Y) = \overline{\{(y, \varphi(y)) \mid y \in Y \setminus X\}} \subseteq Y \times \mathbb{P}^n \qquad$ Blowup of Y along X.

$\quad \pi : Bl_X(Y) \longrightarrow Y$ projection.

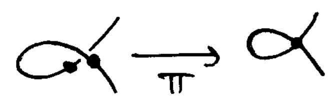
Note: If $Y \setminus X \subseteq Y$ dense open then π surjective.

Point: If Y singular along X, then $Bl_X(Y)$ is often "less singular".

Example $\quad Y = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$. $\quad X = \{0\} \subseteq Y$. $\quad I(X) = (x, y) \subseteq k[Y]$.

$\quad \varphi : Y \setminus \{0\} \longrightarrow \mathbb{P}^1$, $\quad P \longmapsto$ line through 0 and P.

$\quad Bl_X(Y) = \{ (P, \varphi(P)), \; (0, (1:1)), \; (0, (1:-1)) \}$



Fact: A resolution of singularities of any curve can be obtained from iterated blowups.

Note: $\pi^{-1}(X) \subseteq Bl_X(Y)$ is an effective Cartier divisor. (codim $= 1$, locally cut out by 1 fcn.)

In fact, set $\mathcal{L} = \pi_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$, $\qquad \pi_2 : Bl_X(Y) \longrightarrow \mathbb{P}^n$.

$\mathcal{L}$ has global sections $\pi_2^*(z_0), \dots, \pi_2^*(z_n)$, $\qquad z_i \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Set $\quad s = \sum_{i=0}^n f_i \, \pi_2^*(z_i) \in \Gamma(Bl_X(Y), \mathcal{L})$.

Then $\pi^{-1}(X) = Z(s) \subseteq Bl_X(Y)$.

Note: $Bl_X(Y)$ independent of chosen gens. for I:

$Bl_X(Y) \subseteq Y \times \mathbb{P}^n$. $\quad$ Set $J = I(Bl_X(Y)) \subseteq k[Y][z_0, \dots, z_n]$. $\qquad$ graded ideal.

One can recover $Bl_X(Y)$ from $k[Y][z_0, \dots, z_n]/J$.

Claim: $\quad k[Y][z_0, \dots, z_n]/J \cong \bigoplus_{d \geq 0} I^d := \bigoplus I^d \cdot t^d \subseteq k[Y][t]$

$\qquad\qquad = $ subring gen. by $k[Y]$ and $tf_0, \dots, tf_n$.

Morphism $\psi : Y \times \mathbb{A}^1 \longrightarrow Y \times \mathbb{A}^{n+1}$, $\quad (y, t) \longmapsto (y, (tf_0(y), \dots, tf_n(y)))$

$\overline{\psi(Y \times \mathbb{A}^1)} = $ cone over $Bl_X(Y)$. $\quad \Rightarrow \quad J = I(\psi(Y \times \mathbb{A}^1))$.

$\psi^* : k[Y][z_0, \dots, z_n] \longrightarrow k[Y][t]$, $\quad z_i \longmapsto tf_i$

$J = \ker(\psi^*)$.

$\therefore \quad k[Y][z_0, \dots, z_n]/J \cong \operatorname{Image}(\psi^*) = \bigoplus_{d \geq 0} I^d \cdot t^d$.

General varieties

Y any variety, $X \subseteq Y$ closed subvariety.

Take open affine cover $Y = \bigcup Y_i$.

Can glue: $Bl_X(Y) := \bigcup_{glue} Bl_{X \cap Y_i}(Y_i)$

Intrinsic construction:

$I_X \subseteq \mathcal{O}_Y$ ideal sheaf.

Can construct $Bl_X(Y)$ from $\bigoplus_{d \geq 0} I_X^d$, sheaf of $\mathcal{O}_Y$-algebras.