REVIEW OF VARIETIES

1. Affine varieties

$k = \overline{k}$ alg closed field.

R f.g. reduced k-algebra.

Spec-m$(R) = \{$ max. ideals $\mathfrak{m} \subset R$ $\}$

Topology: Zariski closed sets are $Z(I) = \{\mathfrak{m} \supset I\}$

Let $f \in R$. Def. $f : \text{Spec-m}(R) \to k$, $f(\mathfrak{m}) =$ image of f by $R \to R/\mathfrak{m} = k$.

Def: Let $U \subset \text{Spec-m}(R)$ be open, $f : U \to k$ a function.

f is **regular** if it is locally of the form $f(\mathfrak{m}) = p(\mathfrak{m})/q(\mathfrak{m})$, $p, q \in R$.

$\mathcal{O}(U) = \{$ regular $f : U \to k\}$.

Exercise*: $\mathcal{O}(\text{Spec-m}(R)) = R$

Coordinate ring: $A(\text{Spec-m}(R)) = R$ (only for affine varieties)

Example: $R = k[f_1, \dots, f_n] = k[x_1, \dots, x_n]/I$. $(f_1, \dots, f_n) : X \xrightarrow{\sim} Z(I) \subset \mathbb{A}^n$

2. Spaces with functions

Def: A **space with functions** is a top space X with assignment
$U \mapsto \mathcal{O}(U) = \mathcal{O}_X(U) \subset \{$ all fcns $U \to k$ $\}$ (k-subalgebra) such that

(1) $U = \bigcup_\alpha U_\alpha$: $f \in \mathcal{O}_X(U) \Leftrightarrow f|_{U_\alpha} \in \mathcal{O}_X(U_\alpha)\ \forall \alpha$.

(2) $f \in \mathcal{O}_X(U) \Rightarrow D(f) \subset U$ open and $1/f \in \mathcal{O}_X(D(f))$.

Def: A **morphism** of SWFs is a cont. map $\varphi : X \to Y$ such that pullback of regular functions are regular.

I.e. if $V \subset Y$ is open and $f \in \mathcal{O}_Y(V)$, then $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

3. Subspace of SWF

X SWF, $Y \subset X$ any subset. Give Y structure of SWF as follows:

* Subspace topology.

* If $U \subset Y$ is open, $f : U \to k$ function, then f is regular iff f can locally be extended to regular fcn on X.

I.e. $\forall y \in U\ \exists U' \subset X$ and $F \in \mathcal{O}_X(U')$ s.t. $y \in U'$ and $f(x) = F(x)\ \forall x \in U \cap U'$.

Def. A **prevariety** is a SWF X s.t. $\exists$ open cover $X = U_1 \cup \dots \cup U_m$, with $U_i \cong \text{Spec-m}(R_i)$ affine variety for each i.

Exercise: Let $X = \text{Spec-m}(R)$ be affine and $f \in R$. Then $X_f := D(f) \cong \text{Spec-m}(R_f)$.

Exercise: X SWF and Y affine variety.

1-1 correspondence $\{$ morphisms $X \to Y$ $\} \leftrightarrow \{$ k-alg homs $A(Y) \to \mathcal{O}(X)$ $\}$.

Cor: Two affine varieties isomorphic iff coordinate rings isomorphic.

Exercise: $\mathbb{A}^n \smallsetminus \{0\}$ is not affine for $n \geq 2$.

Exercise: An open subset of a prevariety is a prevariety.

Exercise: A closed subset of a prevariety is a prevariety.

Def: X top space. A subset $W \subset X$ is **locally closed** if it is an intersection of an open set and a closed set.

Cor: A locally closed subset of a prevariety is a prevariety.

4. Projective space

Def: $\mathbb{P}^n = (\mathbb{A}^{n+1} \smallsetminus \{0\})/k^* =$ lines through origin in $\mathbb{A}^{n+1}$.

$\pi : \mathbb{A}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ projection.

Topology: $U \subset \mathbb{P}^n$ open $\Leftrightarrow \pi^{-1}(U) \subset \mathbb{A}^{n+1}$ open.

Regular fcns: $f : U \to k$ is regular $\Leftrightarrow \pi^*(f) = f \circ \pi : \pi^{-1}(U) \to k$ regular.

Notation: $(a_0 : \cdots : a_n) = \pi(a_0, \ldots, a_n)$.

Projective coord ring: $\mathcal{O}(\mathbb{A}^{n+1}) = k[x_0, \ldots, x_n]$.

Def: Let $f \in k[x_0, \ldots, x_n]$ homogeneous poly.

$D_+(f) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) \neq 0\}$

Exercise: $D_+(x_i) \cong \mathbb{A}^n$.

Cor: $\mathbb{P}^n = D_+(x_0) \cup \cdots \cup D_+(x_n)$ is a prevariety.

Exercise: X SWF and $\phi : \mathbb{P}^n \to X$ function. Then ϕ is a morphism iff $\phi \circ \pi : \mathbb{A}^{n+1} \smallsetminus \{0\} \to X$ is a morphism.

Def: If $W \subset \mathbb{P}^n$ subset, then $I(W) = I(\pi^{-1}(W)) \subset k[x_0, \ldots, x_n]$.

Def: If $I \subset k[x_0, \ldots, x_n]$ homogeneous ideal, then $Z_+(I) = \pi(Z(I)) \subset \mathbb{P}^n$.

Projective Nullstellensatz: $I \subset k[x_0, \ldots, x_n]$ homogeneous ideal. If $Z_+(I) \neq \emptyset$ then $I(Z_+(I)) = \sqrt{I}$.

5. Projective varieties

Def. A **projective variety** is a closed subset of $\mathbb{P}^n$ (with SWF structure).

A **quasi-projective variety** is a locally closed subset of $\mathbb{P}^n$.

An **affine variety** is a closed subset of $\mathbb{A}^n$.

A **quasi-affine variety** is a locally closed subset of $\mathbb{A}^n$.

Exercise: $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Exercise*: If X is both projective and quasi-affine, then X is finite.

Def: If $X \subset \mathbb{P}^n$ is closed, then proj. coord. ring of X is $k[x_0, \ldots, x_n]/I(X)$. DEPENDS ON EMBEDDING!!

Def: R graded ring, $f \in R_d$.

$R_{(f)} = \{$ homogeneous elts. in R_f of degree zero $\} = \{g/f^m \mid g \in R_{dm}\}$.

Exercise: R f.g. reduced graded k-algebra $\Rightarrow R_{(f)}$ f.g. reduced k-algebra.

Exercise: $X \subset \mathbb{P}^n$ projective, $R = k[x_0, \ldots, x_n]/I(X)$. $f \in R_d$ with $d > 0$. Then $X_f := X \cap D_+(f) \cong \text{Spec-m}(R_{(f)})$.

Hints: Enough to assume $X = \mathbb{P}^n$, $R = k[x_0, \ldots, x_n]$.

Show that $\mathcal{O}(D_+(f)) = R_{(f)}$.

Identity map $R_{(f)} \to \mathcal{O}(D_+(f))$ defines morphism $D_+(f) \to \text{Spec-m}(R_{(f)})$.

Show this is an isomorphism.

6. Products

Let X and Y be SWFs. A **product** of X and Y is a SWF called $X \times Y$ with morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, such that $(X \times Y, \pi_X, \pi_Y)$ is universal.

Exercise: Show that products of SWFs exist and are unique.

Example: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. NOTE: $\mathbb{A}^2$ does not have the product topology!

Exercise: If X and Y are affine varieties, then $X \times Y \cong \text{Spec-m}(A(X) \otimes_k A(Y))$.

Cor: A product of prevarieties is a prevariety.

7. Separated SWFs

Def: A SWF X is **separated** if $\forall$ SWFs Y and morphisms $f, g : Y \to X$ the set $\{y \in Y \mid f(y) = g(y)\} \subset Y$ is closed.

(Algebraic version of Hausdorff.)

Non-example: $X = (\mathbb{A}^1 \smallsetminus \{0\}) \cup \{O_1, O_2\}$ = union of two copies of $\mathbb{A}^1$.

Def: An **algebraic variety** is a separated prevariety.

Exercise: Any subspace of a separated SWF is separated.

Exercise: A product of separated SWFs is separated.

Exercise: $\Delta : X \to X \times X$, $x \mapsto (x, x)$ is a morphism.

Def: $\Delta_X := \Delta(X) \subset X \times X$.

Exercise: $\Delta : X \to \Delta_X$ isomorphism.

Exercise: X is separated $\Leftrightarrow \Delta_X \subset X \times X$ is closed.

Exercise: $\mathbb{A}^n$ is separated, hence all (quasi-) affine varieties are algebraic varieties.

Exercise: $\mathbb{P}^n$ is separated, hence all (quasi-) projective varieties are varieties.

8. Rational maps

Def: A topological space X is **irreducible** if X is not a union of two proper closed subsets.

Let X and Y be irreducible varieties.

Consider pairs (U, f) such that $\emptyset \neq U \subset X$ is open and $f : U \to Y$ is a morphism.

Relation: $(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$.

Exercise: $\sim$ is an equiv. relation. (Since X is irreducible and Y is separated.)

Def: A **rational map** $f : X \dashrightarrow Y$ is an equivalence class for $\sim$.

Exercise: There is a unique maximal open subset of points in X where f is defined as a morphism.

Def: A **rational function** on X is a rational map $f : X \dashrightarrow \mathbb{A}^1 = k$.

f is given by a regular function $f : U \to k$, where $\emptyset \neq U \subset X$ is open.

Def: $k(X) = \{f : X \dashrightarrow k\}$

Exercise: $k(X)$ is a field.

Exercise: $\emptyset \neq U \subset X$ open $\Rightarrow k(U) = k(X)$.

Exercise: X irred. affine variety $\Rightarrow k(X) = K(A(X))$ fraction field.

Def: For $V \subset X$ irred. closed, the **local ring** of X along V is the subring $\mathcal{O}_{X,V} \subset k(X)$ of rational functions that are defined in at least one point of V:

$\mathcal{O}_{X,V} = \{(U, f) \in k(X) \mid U \cap V \neq \emptyset\}$.

Unique max. ideal: $\mathfrak{m}_{X,V} = \{(f, U) \in \mathcal{O}_{X,V} \mid f(x) = 0 \ \forall\, x \in V \cap U\}$.

Exercise: X irred. affine, $V \subset X$ irred. closed $\Rightarrow \mathcal{O}_{X,V} = A(X)_{I(V)}$.

Def: $(U, f) : X \dashrightarrow Y$ is **dominant** if $\overline{f(U)} = Y$.

Exercise: If $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are rational maps and f is dominant, then $\exists$ well-defined composition $g \circ f : X \dashrightarrow Z$.

Exercise: Let X and Y be irreducible varieties. 1-1 correspondence:

$\{$ dominant $f : X \dashrightarrow Y\} \leftrightarrow \{$ field ext. $k(Y) \subset k(X)$ over k $\}$.

Def: $f : X \dashrightarrow Y$ is **birational** if f is dominant and $\exists$ dominant $g : Y \dashrightarrow X$ s.t. $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Def: X and Y are **birationally equivalent** (written $X \approx Y$) iff $\exists$ birational map $f : X \dashrightarrow Y$.

Example: $\mathbb{A}^2 \approx \mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$

Exercise: $X \approx Y \Leftrightarrow k(X) \cong k(Y)$ as k-algebras $\Leftrightarrow$

$\exists$ open subsets $U \subset X$ and $V \subset Y$ s.t. $U \cong V$.

Def: X is **rational** if X is birationally equivalent to $\mathbb{A}^n$ for some n.

Def: X is **unirational** if $\exists$ dominant rational map $f : \mathbb{A}^n \dashrightarrow X$.

Exercise*: $E = Z(y^2 - x^3 + x) \subset \mathbb{A}^2$ is not rational.

Exercise**: If C is a unirational curve, then C is rational.

9. Complete varieties

Def: A variety X is **complete** if for any variety Y, $\pi_Y : X \times Y \to Y$ is closed.

(Analogue of compact manifolds. Schemes: same as proper over $\mathrm{Spec}(k)$.)

Note: 1) Closed subsets of complete varieties are complete.

2) Products of complete varieties are complete.

Example: Points are complete!

Example: $\mathbb{A}^1$ is not complete.

$W = Z(xy - 1) \subset \mathbb{A}^1 \times \mathbb{A}^1$ is closed but $\pi_2(W) = \mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$.

Exercise: Let $\varphi : X \to Y$ be a morphism of varieties. If X is complete then $\varphi(X) \subset Y$ is closed and complete. (Use graph $\Gamma_f \subset X \times Y$.)

Exercise: $\varphi : X \to Y$ cont. map of top. spaces. Then X irred. $\Rightarrow \varphi(X)$ irred.

Cor: If X is irreducible and complete then $\mathcal{O}(X) = k$.

Proof: If $f : X \to \mathbb{A}^1$ is any morphism then $f(X) \subset \mathbb{A}^1$ is closed, complete, and irreducible, hence a point.

Exercise: Any complete quasi-affine variety if finite.

Exercise*: $\mathbb{P}^n$ is complete, hence all projective varieties are complete.