MATH 535 PROBLEM SET 1
DUE WEDNESDAY 9/20 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them.

Problem 1. Show that $I(\mathbb{A}^n) = (0)$.

Problem 2. If $I \subset R$ is any ideal, show that $\sqrt{I}$ is a radical ideal.

Problem 3.
(a) $S \subset I(V(S))$.
(b) $W \subset V(I(W))$.
(c) If W is an algebraic set then $W = V(I(W))$.
(d) If $I \subset k[x_1, \ldots, x_n]$ is any ideal then $V(I) = V(\sqrt{I})$ and $\sqrt{I} \subset I(V(I))$.

Problem 4.
(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$ is closed in $\mathbb{A}^3$ and find $I(X)$.
(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$ of $\mathbb{A}^3$.
(c) Show that $I(Y)$ can't be generated by less than three polynomials.
Hint for (c): Is $I(Y)$ a graded ideal? Are you sure??

Problem 5. Show that $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible closed subset of $\mathbb{A}^3$ and find $I(W)$.

Problem 6. Find $\sqrt{(y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3)}$.

Problem 7. Let X be a Noetherian topological space.
(a) If an irreducible closed set Y is contained in a union $\cup X_i$ of finitely many closed sets X_i, then $Y \subset X_i$ for some i.
(b) X has finitely many components.
(c) X is the union of its components.
(d) X is not the union of any proper subset of its components.

Problem 8. Let X be any space with functions and $Y \subset \mathbb{A}^n$ an affine variety. Show that a function $f : X \to Y$ is a morphism if and only if each coordinate function $f_i : X \to k$ is regular for $1 \leq i \leq n$.

Problem 9. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \to k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.