MATH 535 PROBLEM SET 9
DUE WEDNESDAY 11/15 IN CLASS

Try to solve all of the following problems. Write up at least 4 of them. The first problem is important and merrits another try. The second is also very good to learn from.

Problem 1. [Hartshorne I.5.3 and I.5.4]

Let $X \subset \mathbb{P}^2$ be a curve and $P \in \mathbb{P}^2$ any point. Let $I_{X,P} \subset \mathcal{O}_{\mathbb{P}^2,P}$ be the ideal of functions $f \in \mathcal{O}_{\mathbb{P}^2,P}$ such that $f|_{U \cap X} = 0$ for some open set U containing P. The multiplicity $\mu_P(X)$ of X at P is the largest number r such that $I_{X,P} \subset \mathfrak{m}_P^r$ where $\mathfrak{m}_P \subset \mathcal{O}_{\mathbb{P}^2,P}$ is the maximal ideal.

(a) $P \in X \Leftrightarrow \mu_P(X) \geq 1$.

(b) P is a non-singular point of X iff $\mu_P(X) = 1$.

(c) Let $Y \subset \mathbb{P}^2$ be another curve such that $X \cap Y$ is a finite set. Show that if $P \in X \cap Y$ then $I(X \cdot Y; P) = \dim_k \mathcal{O}_{\mathbb{P}^2,P}/(I_{X,P} + I_{Y,P})$.

(d) $I(X \cdot Y; P) = 1$ iff P is a non-singular point of both X and Y, and the tangent directions at P are different.

(e) $I(X \cdot Y; P) \geq \mu_P(X) \cdot \mu_P(Y)$.

(f) For all but a finite number of lines $L \subset \mathbb{P}^2$ through P we have $\mu_P(X) = I(X \cdot L; P)$.

Hints: (b) If $P = (0,0) \in X \subset \mathbb{A}^2$ and $I(X) = (f) \subset k[x,y]$, what is $\mu_P(X)$?

(c) Assume $P = (0{:}0{:}1) \in \mathbb{P}^2$, $I(X) = (f)$, $I(Y) = (g) \subset S = k[x,y,z]$. Set $Q = I(\{P\}) = (x,y) \subset S$ and $R = \mathcal{O}_{\mathbb{P}^2,P} = k[\frac{x}{z}, \frac{x}{y}]_{(\frac{x}{z}, \frac{y}{z})}$. Then $S_Q = k[x,y,z]_{(y,x)} = R \otimes_k k(z)$, and $\text{length}_{S_Q}(S_Q/(f,g)) = \dim_{k(z)} S_Q/(f,g) = \dim_k R/(I_{X,P} + I_{Y,P})$.

(e) Let $P = (0,0) \in \mathbb{A}^2$, $I(X) = (f)$, $I(Y) = (g) \subset T = k[x,y]$. Set $Q = I(P) = (x,y) \subset T$, $m = \mu_P(X)$, $n = \mu_P(Y)$. The exact sequence $T/Q^n \oplus T/Q^m \xrightarrow{(f,g)} T/Q^{m+n} \to T/(f,g,Q^{m+n}) \to 0$ implies that $\dim_k T_Q/(f,g) \geq mn$.

Problem 2. Let $X \subset \mathbb{P}^5$ be the subset of points $(x_0 : \cdots : x_5)$ such that the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}$$

has rank one. Show that X is a non-singular rational closed subvariety of $\mathbb{P}^5$, and find its dimension and degree.

Hint: $X \cap D_+(x_i) \cong \mathbb{A}^3$. $I(X) = (x_0x_4 - x_1x_3, x_0x_5 - x_2x_3, x_1x_5 - x_2x_4)$. Let $H = V_+(x_0) \subset \mathbb{P}^5$. Then $X \cap H = Z_1 \cup Z_2$ where $Z_1 = V_+(x_0, x_1, x_2)$ and $Z_2 = V_+(x_0, x_3, x_1x_5 - x_2x_4)$. Find $I(X \cdot H; Z_j)$ and $\deg(Z_j)$.

Problem 3. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Show that the map $\text{Hom}(\mathcal{G}, f_*\mathcal{F}) \to \text{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ constructed in class is bijective.

Problem 4. (a) Let X be an affine variety, M a $k[X]$-module, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Show that $\mathrm{Hom}_{k[X]}(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$.

(b) If X is affine and M and N are $k[X]$-modules then $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = (M \otimes_{k[X]} N)^{\sim}$.

(c) If $f : X \to Y$ is a morphism of varieties and $\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_X$-module.

Problem 5. Let X be a variety, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, and $U \subset X$ an open affine subvariety.

(a) $\mathcal{F}|_U \cong \Gamma(U, \mathcal{F})^{\sim}$.

(b) If $\mathcal{F}$ is coherent, then $\Gamma(U, \mathcal{F})$ is a finitely generated $k[U]$-module.

Hint: Reduce to the case $X = U$ is affine with an open affine cover $X = \bigcup V_i$, such that $\mathcal{F}|_{V_i} = \widetilde{M_i}$ for a $k[V_i]$-module M_i. Given $f \in k[X]$ and $s \in \Gamma(X_f, \mathcal{F})$, show that $f^n s$ can be extended to a section in $\Gamma(X, \mathcal{F})$ for some large n. In fact, $\Gamma(X_f, \mathcal{F}) = \Gamma(X, F)_f$, and the $\mathcal{O}_X$-homomorphism $\Gamma(X, \mathcal{F})^{\sim} \to \mathcal{F}$ is an isomorphism.

Problem 6. (a) X is a ringed space, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Then the assignment $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines an $\mathcal{O}_X$-module. It is denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(b) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also invertible and that $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

Problem 7. Let $f : X \to Y$ be a morphism of varieties.

(a) If f is affine, then $f_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_Y$-module.

(b) If f is finite, then $f_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_Y$-module.