MATH 535 PROBLEM SET 11
DUE WEDNESDAY 12/6 IN CLASS

Try to solve all of the following problems. Write up at least 3 of them.

Problem 1. Let $\varphi : X \to Y$ be a finite morphism of nonsingular curves, and let $D \in \mathrm{Div}(Y)$ be a divisor. Let $s = 1 \in \Gamma(V, \mathcal{O}_Y(D))$ be the section given by the constant rational function 1, where $V = Y - \mathrm{Supp}(D)$, and consider the pullback $\varphi^*(s) \in \Gamma(\varphi^{-1}(V), \varphi^*\mathcal{O}_Y(D))$. Show that $\varphi^*(D) = (\varphi^* s) \in \mathrm{Div}(X)$.

Problem 2. (a) Let $X \subset \mathbb{P}^2$ be a non-singular curve of degree 3 and $P \in X$ a point. Show that $\dim_k \Gamma(X, \mathcal{O}_X(n[P])) \geq n$ for all n.

(b) Any proper open subset of X is affine.

[Hint for (a): The section $s = 1$ is in $\Gamma(X, \mathcal{O}_X(n[P]))$ for all n. Start by finding a quotient f/g of linear forms on $\mathbb{P}^2$, which has at worst a double pole at P, and no other poles on X. Pay attention to the lines $V_+(f)$ and $V_+(g)$.]

Problem 3. (a) Let $F, G, H \in k[x, y, z]$ be forms such that $V_+(G, H, z) = \emptyset$ in $\mathbb{P}^2$. Show that if $zF \in (G, H)$ then $F \in (G, H)$. [Hint: Use that $G_0 = G(x, y, 0)$ and $H_0 = H(x, y, 0)$ are relatively prime.]

(b) Let $C \subset \mathbb{P}^2$ be a curve, and set $\mathcal{O}_C(n) = \mathcal{O}_{\mathbb{P}^2}(n)|_C$. Then $\Gamma(C, \mathcal{O}_C(n)) = (k[x, y, z]/I(C))_n$ for all $n \geq 0$. [Hint: If $C = V_+(H) \subset D_+(y) \cup D_+(z)$ and if σ is a global section of $\mathcal{O}_C(n)$ then $\sigma/y^n = F(x, y, z)/y^m$ and $\sigma/z^n = A(x, y, z)/z^m$ for forms $F, A \in k[x, y, z]$ of degree $m \geq n$. Now use part (a).]

(c) Define the *arithmetic genus* of C to be $1 - P_C(0)$ where $P_C(m)$ is the Hilbert polynomial of $C \subset \mathbb{P}^2$. Show that $p_a = \frac{(d-1)(d-2)}{2}$ where d is the degree of C and that $\dim_k \Gamma(C, \mathcal{O}_C(n)) = nd + 1 - p_a$ for all large integers n.

Problem 4. (a) Let $C \subset \mathbb{P}^2$ be a non-singular curve and $Y \subset \mathbb{P}^2$ an irreducible curve different from C. Set $Y.C = \sum_P I(Y \cdot C; P)\, P \in \mathrm{Div}(C)$. Show that $\mathcal{L}([Y])|_C \cong \mathcal{L}(Y.C)$ on C.

(b) Let $L = V_+(f)$ and $M = V_+(g) \subset \mathbb{P}^2$ be lines (not equal to C) where $f, g \in k[x, y, z]$ are linear forms. Then the divisor of $f/g \in k(C)$ is $(f/g) = L.C - M.C$.

Problem 5.

(a) If D is any Weil divisor of degree m on $\mathbb{P}^1$, then $\mathcal{O}_{\mathbb{P}^1}(D) \cong \mathcal{O}_{\mathbb{P}^1}(m)$.

(b) Let $E \subset \mathbb{P}^2$ be a closed non-singular curve of degree 3. Show that for some divisor $D \in \mathrm{Div}(C)$ of degree 3, we have $\dim_k \Gamma(C, \mathcal{O}_C(D)) = 3$. [Hint: Take $D = E.L$ where $L \subset \mathbb{P}^2$ is a line.]

(c) Prove that E is not rational.