ALGEBRAIC GEOMETRY I, PROBLEM SET 1

Some problems are easy, others are trickier. You are encouraged to collaborate about the hard ones. Remember, serious people should do all these problems, ask for hints if required. (Later problem sets will have fewer problems.) Have fun!

Problem 1. Show that $I(\mathbb{A}^n) = (0)$.

Problem 2. If $I \subset R$ is any ideal, show that $\sqrt{I}$ is a radical ideal.

Problem 3. (a) $S \subset I(V(S))$.
(b) $W \subset V(I(W))$.
(c) If W is an algebraic set then $W = V(I(W))$.
(d) If $I \subset k[x_1, \ldots, x_n]$ is any ideal then $V(I) = V(\sqrt{I})$ and $\sqrt{I} \subset I(V(I))$.

Problem 4. [Hartshorne I.1.2 and I.1.11]
(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$ is closed in $\mathbb{A}^3$ and find $I(X)$.
(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$ of $\mathbb{A}^3$.
(c) Show that $I(Y)$ can't be generated by less than three polynomials.
Hint: Is $I(Y)$ a graded ideal? Are you sure??

Problem 5. Let R be a commutative ring. The following are equivalent:
(a) R is Noetherian.
(b) Every ascending chain of ideals in R stabilizes.
(c) Every non-empty collection of ideals of R has a maximal element.

Problem 6. Show that $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible closed subset of $\mathbb{A}^3$ and find $I(W)$.
Hint: Construct a homomorphism $k[x, y, z] \to k[T]$ with kernel $I(W)$.

Problem 7. Find $\sqrt{(y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3)}$.

Problem 8. Let X be a Noetherian topological space.
(a) If an irreducible closed set Y is contained in a union $\cup X_i$ of finitely many closed sets X_i, then $Y \subset X_i$ for some i.
(b) X has finitely many components.
(c) X is the union of its components.
(d) X is not the union of any proper subset of its components.

Problem 9. Let X be any space with functions and $Y \subset \mathbb{A}^n$ an affine variety. Show that a function $f : X \to Y$ is a morphism if and only if each coordinate function $f_i : X \to k$ is regular for $1 \le i \le n$.

Problem 10. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \to k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Problem 11. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \to k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.

Problem 12. (a) $k[\mathbb{A}^n \smallsetminus \{0\}] = k[x_1, \dots, x_n]$ for $n \geq 2$.
(b) $\mathbb{A}^n \smallsetminus \{0\}$ is not an affine variety for $n \geq 2$.
(c) Every global regular function on $\mathbb{P}^n$ is constant, i.e. $k[\mathbb{P}^n] = k$.
(d) $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Problem 13. Let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Problem 14. Let $X \subset \mathbb{A}^n$ be a closed subvariety. Identify $\mathbb{A}^n$ with $D_+(x_0) \subset \mathbb{P}^n$ and let $\bar{X}$ be the closure of X in $\mathbb{P}^n$. Show that $I(\bar{X}) = I(X)^* \subset k[x_0, \dots, x_n]$. ($I(X)^*$ is defined in the notes for 9/18.)

Problem 15. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Problem 16. Show that if R is a finitely generated reduced k-algebra then the space with functions Spec-m(R) is an affine variety.

Problem 17. Let X be any space with functions. A map $\varphi : \mathbb{P}^n \to X$ is a morphism if and only if $\varphi \circ \pi : \mathbb{A}^{n+1} \smallsetminus \{0\} \to X$ is a morphism.

Problem 18. Let $\varphi : X \to Y$ be a morphism of spaces with functions and suppose $Y = \bigcup V_i$ is an open covering such that each restriction $\varphi : \varphi^{-1}(V_i) \to V_i$ is an isomorphism. Then φ is an isomorphism.

Problem 19. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x, y, z]$ of degree 2, then $C \cong \mathbb{P}^1$.

Problem 20. Let X and Y be spaces with functions and let (P, π_X, π_Y) and (P', π'_X, π'_Y) be two products of X and Y. Show that there is a unique isomorphism $\varphi : P \xrightarrow{\sim} P'$ such that $\pi_X = \pi'_X \circ \varphi$ and $\pi_Y = \pi'_Y \circ \varphi$.