ALGEBRAIC GEOMETRY I, PROBLEM SET 2

Problem 1. Prove that the Segre map $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ gives an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a closed subvariety of P^N, where $N = nm + n + m$.

Problem 2. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 3. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y.
(a) Show that X is separated.
(b) Show that $\mathbb{P}^n$ has this property.

Problem 4. [Hartshorne II.2.16 and II.2.17]
Let X be any variety and $f \in k[X]$ a regular function.
(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i.]
(b) $k[D(f)] = k[X]_f$.
(c) Let R be a k-algebra and let $f_1, \ldots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \ldots, f_r) = R$. If R_{f_i} is a finitely generated k-algebra for each i, then R is a finitely generated k-algebra.
(d) Suppose $f_1, \ldots, f_r \in k[X]$ satisfy $(f_1, \ldots, f_r) = k[X]$ and $D(f_i)$ is affine for each i. Then X is affine.

Problem 5. Let E be the elliptic curve $V_+(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x : y : z) = (x : z)$ and $g(x : y : z) = (y : z)$. (These are just projections to the x and y axis on the open subset $D_+(z)$.)
(a) Find the maximal open sets in E where f and g are defined as morphisms.
(b) Find the degrees of the field extensions $k(t) \subset k(E)$ induced by f and g.
(c) Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point. (Part of the exercise is to define what "typical" means.)

Problem 6. Let X be a projective variety and $\varphi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that φ is defined as a morphism on all of $\mathbb{P}^1$.

Problem 7. (a) If X has components $X_1, \ldots, X_m$ then $\dim(X) = \max \dim(X_i)$.
(b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 8. The commutative algebra result *lying over* states that if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.
(a) Use lying over to show that if $\varphi : X \to Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of Y.
(b) If $\varphi : X \to Y$ is any morphism of varieties, then its image $\varphi(X)$ is *constructible*, i.e. a finite union of locally closed subsets of Y.

Problem 9. [Hartshorne I.5.2]
Assume $\text{char}(k) \neq 2$. Locate the singular points of the surfaces $X = V(xy^2 - z^2)$, $Y = V(x^2 + y^2 - z^2)$, and $Z = V(xy + x^3 + y^3)$ in $\mathbb{A}^3$. (Take a look at the nice pictures in Hartshorne!)

Problem 10. Assume $\text{char}(k) = 0$. Let $X = V_+(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \dots, x_n]$.
(a) Show that $X_{\text{sing}} = V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$.
(b) Show that $X_{\text{sing}} \neq X$.

Problem 11. [Shafarevich II.1.13]
(a) Show that an intersection of r hypersurfaces in $\mathbb{P}^r$ is never empty.
(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that X contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq n/2$. Prove that X is singular. [Hint: Choose the coordinates on $\mathbb{P}^n$ such that $L = V_+(x_{r+1}, x_{r+2}, \dots, x_n) \subset \mathbb{P}^n$.]

Problem 12. [Shafarevich II.1.10].
Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If X has two different singular points, then X contains the line joining them.

Problem 13. If X is a variety and $x \in X$, we define the Zariski cotangent space to X at x to be $\mathfrak{m}_x/\mathfrak{m}_x^2$. The Zariski tangent space is the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Show that if $f : X \to Y$ is a morphism of varieties with $f(x) = y$, then f induces linear maps $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ and $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$.

Problem 14. [Mostly Hartshorne I.6.3]
Give examples of varieties X and Y, a point $P \in X$, and a morphism $\varphi : X \smallsetminus \{P\} \to Y$ such that φ can't be extended to a morphism on all of X in each of the cases:
(a) X is a non-singular curve and Y is not projective.
(b) X is a curve, P is a singular point on X, Y is projective.
(c) X is non-singular of dimension at least two, Y is projective.

Problem 15. Let X and Y be curves and $\varphi : X \to Y$ a birational morphism.
(a) X_{sing} is a proper closed subset of X.
(b) $\varphi(X_{\text{sing}}) \subset Y_{\text{sing}}$.
(c) If $y \in Y$ is a non-singular point, then $\varphi^{-1}(y)$ contains at most one point.

Problem 16. Two non-singular projective curves are isomorphic if and only if they have the same function field.

Problem 17. Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$. Show that if $P \in E$ is any point then $E \smallsetminus \{P\}$ is affine.

Problem 18. [Hartshorne I.6.2]
Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$, $\text{char}(k) \neq 2$.
(a) E is a non-singular curve.
(b) The units in $k[E]$ are the non-zero elements of k. [Hints: Define an automorphism $\sigma : k[E] \to k[E]$ fixing x and sending y to $-y$. Then define a norm $N : k[E] \to k[x]$ by $N(a) = a\,\sigma(a)$. Show that $N(1) = 1$ and $N(ab) = N(a)N(b)$.]
(c) $k[E]$ is not a unique factorization domain.
(d) Show that E is not rational.

Problem 19. Let $m_0, m_1, \ldots, m_N \in k[x_0, \ldots, x_n]$ be all the monomials of degree d. The *Veronese embedding* is the map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by

$$v_d(x_0 : \cdots : x_n) = (m_0(x_0, \ldots, x_n) : \cdots : m_N(x_0, \ldots, x_n)) .$$

(a) Show that v_d is an isomorphism of $\mathbb{P}^n$ with a closed subvariety in $\mathbb{P}^N$.

(b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $S = V_+(f)$ where $f \in k[x_0, \ldots, x_n]$ is a form of degree d. Show that $S = {v_d}^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

Problem 20. Let L_1, L_2, and L_3 be lines in $\mathbb{P}^3$ such that none of them meet.

(a) There exists a unique quadric surface $S \subset \mathbb{P}^3$ containing L_1, L_2, and L_3. [Hint: Start by applying an automorphism of $\mathbb{P}^3$ to make the lines nice.]

(b) S is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting L_1, L_2, and L_3.

(c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet L_1, L_2, or L_3. Then the number of lines meeting L_1, L_2, L_3, and L_4 is equal to the number of points in $L_4 \cap S$, which is one, two, or infinitely many.