ALGEBRAIC GEOMETRY I, PROBLEM SET 4

Problem 1. [Mostly Hartshorne I.7.1] In this problem, just find the numbers and give an argument why they are correct that *could* be expanded into a proof.

(a) Find the degree of $v_d(\mathbb{P}^n)$ in $\mathbb{P}^N$ where v_d is the Veronese embedding.

(b) Find the degree of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$.

(c) Challenge: Find the degree of $\mathrm{Gr}(2,5)$ in $\mathbb{P}^9$.

Problem 2. Let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be any non-constant morphism. Then $\dim \varphi(\mathbb{P}^n) = n$. Furthermore, φ is the composition of a Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$, a projection $\mathbb{P}(k^N) - \mathbb{P}(L) \to \mathbb{P}(k^N/L)$ for some linear subspace $L \subset k^N$, and an inclusion of a linear subspace $\mathbb{P}(k^N/L) \subset \mathbb{P}^m$.

Problem 3. (a) Let $\varphi : X \to Y$ be an affine morphism of pre-varieties. Show that if Y is separated then so is X.

(b) X is an irreducible affine variety, $U \subset X$ an open affine subset, $\bar{U} \subset \bar{X}$ their normalizations, and $\pi : \bar{X} \to X$ the normalization map. Show that $\pi^{-1}(U) = \bar{U}$.

(c) If X is any irreducible variety then $\pi : \bar{X} \to X$ is a finite morphism. Conclude that $\bar{X}$ is separated.

Problem 4. (a) If Y is a normal variety and $f : Y \to X$ a dominant morphism, then there exists a unique morphism $\bar{f} : Y \to \bar{X}$ such that $f = \pi \circ \bar{f}$.

(b) Give a counter example to (a) when f is not dominant.

Problem 5. $X = V(xy - z^2) \subset \mathbb{A}^3$ is normal. [Hint: $k[X] = k[x, xt, xt^2] \subset k(x,t)$ where $t = z/x$.]

Problem 6. If X is any normal rational variety then $\mathrm{C}\ell(X)$ is a finitely generated Abelian group.

Problem 7. (a) Let $F, G, H \in k[x,y,z]$ be forms such that $V_+(G,H,z) = \emptyset$ in $\mathbb{P}^2$. Show that if $zF \in (G,H)$ then $F \in (G,H)$. [Hint: Use that $G_0 = G(x,y,0)$ and $H_0 = H(x,y,0)$ are relatively prime.]

(b) Let $C \subset \mathbb{P}^2$ be a curve, and set $\mathcal{O}_C(n) = \mathcal{O}_{\mathbb{P}^2}(n)|_C$. Then $\Gamma(C, \mathcal{O}_C(n)) = (k[x,y,z]/I(C))_n$ for all $n \geq 0$. [Hint: If $C = V_+(H) \subset D_+(y) \cup D_+(z)$ and if σ is a global section of $\mathcal{O}_C(n)$ then $\sigma/y^n = F(x,y,z)/y^m$ and $\sigma/z^n = A(x,y,z)/z^m$ for forms $F, A \in k[x,y,z]$ of degree $m \geq n$. Now use part (a).]

(c) Define the *arithmetic genus* of C to be $1 - P_C(0)$ where $P_C(m)$ is the Hilbert polynomial of $C \subset \mathbb{P}^2$. Show that $p_a = \frac{(d-1)(d-2)}{2}$ where d is the degree of C and that $\dim_k \Gamma(C, \mathcal{O}_C(n)) = nd + 1 - p_a$ for all large integers n.