ALGEBRAIC GEOMETRY I, PROBLEM SET 5

Problem 1. (a) Let $C \subset \mathbb{P}^2$ be a non-singular curve and $Y \subset \mathbb{P}^2$ an irreducible curve different from C. Set $Y.C = \sum_P I(Y \cdot C; P)\, P \in \mathrm{Div}(C)$. Show that $\mathcal{L}([Y])|_C \cong \mathcal{L}(Y.C)$ on C.

(b) Let $L = V_+(f)$ and $M = V_+(g) \subset \mathbb{P}^2$ be lines (not equal to C) where $f, g \in k[x, y, z]$ are linear forms. Then the divisor of $f/g \in k(C)$ is $(f/g) = L.C - M.C$.

Problem 2. Let $E \subset \mathbb{P}^2$ be an elliptic curve and $P_0 \in E$ any point. Show that the map $E \to \mathrm{C}\ell^\circ(E)$ given by $P \mapsto P - P_0$ is bijective.

Problem 3. Let $D : S \to M$ be an R-derivation and $p(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ a polynomial. Then $D(p(a_1, \dots, a_n)) = \sum_{i=1}^n \frac{\partial p}{\partial x_i}(a_1, \dots, a_n) D(a_i)$ for all elements $a_1, \dots, a_n \in S$.

Problem 4. Let $E = V_+(zy^2 - x^3 + z^2 x) \subset \mathbb{P}^2$, $\mathrm{char}(k) \neq 2$. Show that $\Omega_E \cong \mathcal{O}_E$. [Hint: Compute the divisor of the section $d(x/z)$ of Ω_E.]

Problem 5. Let $\mathrm{GL}_n(k)$ act on $\mathrm{Gr}(d, n)$ by $g.V = \{g(x) \mid x \in V\}$. Show that for any points $V_1, V_2 \in \mathrm{Gr}(d, n)$ there exists an element $g \in \mathrm{GL}_n(k)$ such that $g.V_1$ and $g.V_2$ are both in $U_{\{1,\dots,d\}} \subset \mathrm{Gr}(d, n)$. Conclude that $\mathrm{Gr}(d, n)$ is separated.

Problem 6. (a) Let $0 < p < q < n$ be integers and $E = k^n$. Show that the set $\{(V, W) \in \mathrm{Gr}(p, E) \times \mathrm{Gr}(q, E) \mid V \subset W\}$ is closed in $\mathrm{Gr}(p, E) \times \mathrm{Gr}(q, E)$.

(b) Let $0 < d_1 < d_2 < \cdots < d_m < n$ be integers and let $\mathrm{F}\ell(d_1, \dots, d_m; E)$ be the set of flags of subspaces $V_1 \subset V_2 \subset \cdots \subset V_m \subset E$ such that $\dim V_i = d_i$. Give this set a structure of projective variety.

Problem 7. Set $E = k^n$, $X = \mathrm{Gr}(d, E)$, and let $F_1 \subset F_2 \subset \cdots \subset F_n = E$ be a flag of subspaces such that $\dim F_i = i$. Given a sequence of integers $a = (0 < a_1 < a_2 < \cdots < a_d \leq n)$, let $\Omega^\circ_a(F_\bullet)$ be the set of all $V \in X$ such that $\dim(V \cap F_p) = i$ whenever $a_i \leq p < a_{i+1}$, $0 \leq i \leq d$. (We set $a_0 = 0$ and $a_{d+1} = n + 1$.)

(a) Show that $\Omega^\circ_a(F_\bullet) \cong \mathbb{A}^m$ where $m = \sum a_i - \binom{d+1}{2}$.

(b) Show that the orbits for the action of the upper triangular matrices on X are the sets $\Omega^\circ_a(F_\bullet)$ for all sequences a where $F_i = \mathrm{span}\{e_1, \dots, e_i\}$.

(c) The *Schubert varieties* in X are the closures $\Omega_a(F_\bullet) = \overline{\Omega^\circ_a(F_\bullet)}$. Find a singular Schubert variety in some Grassmannian.