ALGEBRAIC GEOMETRY I, PROBLEM SET 1 SOLUTIONS

Problem 1. Show that $I(\mathbb{A}^n) = (0)$.

Solution: Start by showing that any algebraically closed field k is infinite, e.g. by showing that there are infinitely many monic irreducible polynomials in $k[x]$. Then use induction on n and the fact that if k is infinite and $0 \neq f \in k[x]$ then $f(a) \neq 0$ for some $a \in k$.

Problem 4. [Hartshorne I.1.2 and I.1.11]

(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$ is closed in $\mathbb{A}^3$ and find $I(X)$.

(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$ of $\mathbb{A}^3$.

(c) Show that $I(Y)$ can't be generated by less than three polynomials.

Hint: Is $I(Y)$ a graded ideal? Are you sure??

Solution: (b) Let $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$. Set $I = (y^2 - xz, yz - x^3, z^2 - x^2y) \subset k[x, y, z]$. I claim that $Y = V(I)$.

Since the generators of Y vanish on the points of Y we have $Y \subset V(I)$. Let $p = (x, y, z) \in V(I)$. If $x = 0$ then $y = z = 0$ as well, so $p \in Y$. If $x \neq 0$, set $t = y/x$ and notice that

$$t^3 = y^3/x^3 = y\,xz/x^3 = yz/x^2 = x^3/x^2 = x$$
$$t^4 = tx = y$$
$$t^5 = ty = y^2/x = xz/x = z$$

This shows that $p = (x, y, z) = (t^3, t^4, t^5) \in Y$.

Now let $f \in k[x, y, z]$ be any polynomial. Notice that if we replace any occurrence of y^2 in f by xz, the resulting polynomial has the same residue as f modulo I. Similarly yz can be replaced with x^3 and z^2 can be replaced with x^2y. Keep replacing monomials in this way until we reach a polynomial of the form $h = h_1(x) + y\,h_2(x) + z\,h_3(x)$ with $h_1, h_2, h_3 \in k[x]$, such that $h \equiv f$ modulo I. (Notice that the process of replacing monomials has to stop since the number of y's and z's in each monomial decreases at each step.)

I claim that $I(Y) = I$. We already know that $I \subset I(Y)$, so let $f \in I(Y)$. Then choose $h = h_1(x) + yh_2(x) + zh_3(x)$ as above. Then we have

$$h_1(t^3) + t^4\,h_2(t^3) + t^5\,h_3(t^3) = h(t^3, t^4, t^5) = f(t^3, t^4, t^5) = 0$$

for all $t \in k$ which implies that $h_1 = h_2 = h_3 = 0$ by comparing powers of t. We conclude that $f \equiv 0$ modulo I so $f \in I$.

(c) Define a grading of $k[x, y, z]$ by taking the degrees of x, y, and z to be 3, 4, and 5, respectively. Then the generators of I are homogeneous of degrees 8, 9, and 10. Now suppose $I = (f, g)$ is generated by two polynomials. Then we can write

$$y^2 - xz = p_1\,f + p_2\,g$$

for some $p_1, p_2 \in k[x, y, z]$. Since this equation must still hold if we clear all terms of degree above 10, we get

$$y^2 - xz = \alpha_1\,\tilde{f} + \alpha_2\,\tilde{g}$$

where $\tilde{f}$ and $\tilde{g}$ are the sums of the terms of degrees at most 10 in f and g, and $\alpha_i = p_i(0,0,0) \in k$. Here it is crucial that the degree of any variable times f or g has all terms of degree at least 11. Similarly we can write

$$yz - x^3 = \beta_1\, \tilde{f} + \beta_2\, \tilde{g} \qquad \text{and} \qquad z^3 - x^2 y = \gamma_1\, \tilde{f} + \gamma_2\, \tilde{g}$$

for some $\beta_i, \gamma_i \in k$. But now the three dimensional vector space spanned by the generators of I is spanned by two elements, which is impossible.

Problem 6. Show that $W = \{(x,y,z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible closed subset of $\mathbb{A}^3$ and find $I(W)$.

Hint: Construct a homomorphism $k[x,y,z] \to k[T]$ with kernel $I(W)$.

Solution: Define a k-algebra homomorphism $\alpha : k[x,y,z] \to k[t]$ by $\alpha(x) = t^9$, $\alpha(y) = t^6$, and $\alpha(z) = t^4$. Then $I = (x^2 - y^3, y^2 - z^3) \subset \ker(\alpha)$. If $f \in k[x,y,z]$ is any polynomial we can write $f = g_1(z) + g_2(z)x + g_3(z)y + g_4(z)xy + h$ where $g_i \in k[z]$ and $h \in I$, and $\alpha(f) = g_1(t^4) + g_2(t^4)t^9 + g_3(t^4)t^6 + g_4(t^4)t^{15}$. If $\alpha(f) = 0$ then $g_1 = g_2 = g_3 = g_4 = 0$, so $f = h \in I$. This shows that $I = \ker(\alpha)$. In particular $k[x,y,z]/I$ is a subring of the domain $k[t]$, so I must be a prime ideal. From this it follows that $W = V(I)$ is irreducible and $I(W) = I$.

Problem 7. Find $\sqrt{(y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3)}$.

Solution: $\sqrt{I} = I(V(I))$.

Problem 10. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \to k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Solution: Let $R = k[X] = k[x,y,z,w]/(xy - zw)$. It is enough to show that if $q \in R$ is any regular function such that $q(u) \neq 0$ for all $u \in U$ then q is constant. Since the restriction of q to $D(w) \subset X$ is a non-zero regular function, q is a *unit* in $k[D(w)] = R_w = k[x,y,w]_w$. (The identification of R_w with $k[x,y,w]_w$ is by mapping z to xy/w.) The only units in this ring are the elements of the form αw^n where $\alpha \in k^*$ and $n \in \mathbb{Z}$. If $n < 0$ we can replace q with q^{-1}, so assume $n \geq 0$. Then since $(0,1,0,0) \in D(y) \subset U$ we must have $q(0,1,0,0) \neq 0$, so $n = 0$.

Problem 11. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \to k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.

Solution: If f, g, p, q are elements of a UFD R such that $\gcd(f,g) = \gcd(p,q) = 1$ and $f/g = p/q \in R_0$ then $f = p$ and $g = q$ up to units.

Problem 12. (a) $k[\mathbb{A}^n \smallsetminus \{0\}] = k[x_1, \dots, x_n]$ for $n \geq 2$.

(b) $\mathbb{A}^n \smallsetminus \{0\}$ is not an affine variety for $n \geq 2$.

(c) Every global regular function on $\mathbb{P}^n$ is constant, i.e. $k[\mathbb{P}^n] = k$.

(d) $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Solution: (a) Any regular function on $U = \mathbb{A}^n \smallsetminus \{0\}$ can be written as f/g where $f, g \in k[\mathbb{A}^n]$ and $g \neq 0$ on U. This implies that g is constant.

(b) The inclusion $U \subset \mathbb{A}^n$ corresponds to the identity map $k[\mathbb{A}^n] = k[\mathbb{A}^n]$, so if U was affine, then this inclusion would have to be an isomorphism.

(c) The regular functions of degree zero in $k[\mathbb{A}^{n+1} \smallsetminus \{0\}]$ are all constant by (a).

(d) If $\mathbb{P}^n$ is quasi-affine then there exists an injective morphism $\varphi : \mathbb{P}^n \to \mathbb{A}^m$ for some m. Since each coordinate function φ_i is regular, it must be constant, so $\varphi(\mathbb{P}^n)$ is a single point.

Problem 13. Let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Solution: The k-algebra homomorphism $\varphi^* : k[x,y]/(y^2 - x^3) \to k[t]$ is defined by $\varphi^*(x) = t^2$ and $\varphi^*(y) = t^3$. Since this map is not surjective, φ is not an isomorphism.

Problem 15. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Solution: Let $F \in k[x_0, \dots, x_n]$ be a representative for f. Then we know that $D_+(F) \subset \mathbb{P}^n$ is affine, which implies that the closed subset $D_+(f) = D_+(F) \cap X$ of $D_+(F)$ is affine as well. This construction can also be used to compute the affine coordinate ring of $D_+(f)$.

Problem 19. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x,y,z]$ of degree 2, then $C \cong \mathbb{P}^1$.

Solution: Write $f(x,y,z) = a\,x^2 + b\,xy + c\,xz + p\,y^2 + q\,yz + r\,z^2 \in k[x,y,z]$ where $a, b, c, p, q, r \in k$. Then $f(x,y,z) = (x,y,z) \cdot A \cdot (x,y,z)^T$ where A is the matrix

$$A = \begin{bmatrix} a & b/2 & c/2 \\ b/2 & p & q/2 \\ c/2 & q/2 & r \end{bmatrix}.$$

Since A is symmetric and $\operatorname{char}(k) \neq 2$, we can write $BAB^T = \operatorname{diag}(d_1, d_2, d_3)$ for some invertible matrix B (check this!). If we use B to change coordinates on $\mathbb{P}^2$ we may therefore assume that $f(x,y,z) = d_1\,x^2 + d_2\,y^2 + d_3\,z^2$. Since f is irreducible, all d_i must be non-zero, so by further scaling the variables we may assume $f = x^2 + y^2 + z^2$. Finally, we can define an explicit isomorphism $\varphi : \mathbb{P}^1 \xrightarrow{\sim} V_+(x^2 + y^2 + z^2) \subset \mathbb{P}^2$ by $\varphi(s:t) = (2st : s^2 - t^2 : \sqrt{-1}(s^2 + t^2))$.