ALGEBRAIC GEOMETRY I, PROBLEM SET 2 SOLUTIONS

Problem 1. Prove that the Segre map $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ gives an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a closed subvariety of P^N, where $N = nm + n + m$.

Solution: Denote the variables on $\mathbb{P}^N$ by z_{ij} for $0 \le i \le n$ and $0 \le j \le m$. Set $I = (z_{ij}z_{kl} - z_{il}z_{kj}) \subset k[z_{ij}]$. Then $s(\mathbb{P}^n \times \mathbb{P}^m) \subset V_+(I) \subset \mathbb{P}^N$. Given fixed indexes i, j, define $\varphi : V_+(I) \cap D_+(z_{ij}) \to \mathbb{P}^n \times \mathbb{P}^m$ by $\varphi((z_{kl})) = (z_{0j} : z_{1j} : \cdots : z_{nj}) \times (z_{i0} : \cdots : z_{im})$. These maps are compatible and define a morphism $\varphi : V_+(I) \to P^n \times \mathbb{P}^m$ which is inverse to the Segre map.

Problem 3. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y.

(a) Show that X is separated.

(b) Show that $\mathbb{P}^n$ has this property.

Solution: (a) Given two morphisms $f, g : Y \to X$ we show that $D = \{y \in Y \mid f(y) \neq g(y)\}$ is open in Y. If $f(y) \neq g(y)$ then take an open affine $U \subset X$ such that $f(y), g(y) \in U$. Then $V = f^{-1}(U) \cap g^{-1}(U) \subset X$ is open and f and g restrict to morphisms $f, g : V \to U$. Since U is separated, it follows that $\{v \in V \mid f(v) \neq g(v)\}$ is open in Y. Finally, D is the union of such sets.

Problem 4. [Hartshorne II.2.16 and II.2.17]

Let X be any variety and $f \in k[X]$ a regular function.

(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i.]

(b) $k[D(f)] = k[X]_f$.

(c) Let R be a k-algebra and let $f_1, \ldots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \ldots, f_r) = R$. If R_{f_i} is a finitely generated k-algebra for each i, then R is a finitely generated k-algebra.

(d) Suppose $f_1, \ldots, f_r \in k[X]$ satisfy $(f_1, \ldots, f_r) = k[X]$ and $D(f_i)$ is affine for each i. Then X is affine.

Solution: (a) If f is a regular function on a variety X we will write $X_f = D(f)$. If h is regular on X_f and $X = U_1 \cup \cdots \cup U_m$ where each U_i is open affine, then for each i we have $h \in k[(U_i)_f] = k[U_i]_f$, so $h = g/f^n$ for some $g \in k[U_i]$, i.e. $f^n h = g \in k[U_i]$. If we take n large enough to work for all $1 \le i \le m$, we obtain $f^n h \in k[X]$.

(b) Using (a) it is easy to check that the obvious k-algebra homomorphism $k[X]_f \to k[X_f]$ is an isomorphism.

(c) We first prove that $k[X]$ is a finitely generated k-algebra. By the assumptions we can write $g_1 f_1 + \cdots + g_r f_r = 1$ with $g_i \in k[X]$. Since X_{f_i} is affine, we know that $k[X]_{f_i} = k[X_{f_i}]$ is a finitely generated k-algebra for each i. Choose $h_{i,1}, \ldots, h_{i,N} \in k[X]$ so that $k[X]_{f_i}$ is generated by $h_{i,1}, \ldots, h_{i,N}, f_i^{-1}$. We claim that $k[X]$ is generated by the elements f_i, g_i, $h_{i,j}$ for $1 \le i \le r$ and $1 \le j \le N$. In fact, if

$p \in k[X]$ is any regular function, then $f_i^M p \in k[f_i, h_{i,1}, \dots, h_{i,N}] \subset k[X]$ for M sufficiently large. It follows that $p = (g_1 f_1 + \cdots + g_r f_r)^{rM} p \in k[f_i, g_i, h_{i,j}]$.

Set $Y = \text{Spec-m}(k[X])$ and let $\varphi : X \to Y$ be the morphism given by the identity $k[X] \to k[X]$. Since $(f_1, \dots, f_r)$ is the unit ideal it follows that $Y = Y_{f_1} \cup \cdots \cup Y_{f_r}$ is an open covering. Since X_{f_i} is affine and since $k[X_{f_i}] = k[Y_{f_i}]$ by (b) we deduce that φ restricts to an isomorphism $\varphi : X_{f_i} = \varphi^{-1}(Y_{f_i}) \to Y_{f_i}$ for all i. It follows that φ is an isomorphism.

Problem 5. Let E be the elliptic curve $V_+(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x : y : z) = (x : z)$ and $g(x : y : z) = (y : z)$. (These are just projections to the x and y axis on the open subset $D_+(z)$.)

(a) Find the maximal open sets in E where f and g are defined as morphisms.

(b) Find the degrees of the field extensions $k(t) \subset k(E)$ induced by f and g.

(c) Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point. (Part of the exercise is to define what "typical" means.)

Solution: (a) Both f and g can be defined on all of E, since
$f(x : y : z) = (x : z) = (x^3 : x^2z) = (y^2z + xz^2 : x^2z) = (y^2 + xz : x^2)$.

(b) E has an open subset isomorphic to $V(y^2 - x^3 + x) \subset \mathbb{A}^2$, so $k(E)$ is the field of fractions of $k[x, y]/(y^2 - x^3 + x)$. Now f^* is the map $k(t) \to k(E)$ given by $t \mapsto x$. $k(E)$ has the basis $\{1, y\}$ over $k(x)$, so the degree of the extension $f^* : k(t) \subset k(E)$ is two. Similarly g^* is the map $k(t) \to k(E)$ given by $t \mapsto y$; $k(E)$ has the basis $\{1, x, x^2\}$ over $k(y)$ so the degree is 3.

Problem 6. Let X be a projective variety and $\varphi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that φ is defined as a morphism on all of $\mathbb{P}^1$.

Solution: It is enough to show that if $U \subset \mathbb{A}^1$ is a non-empty open set such that $0 \notin U$ then any morphism $f : U \to \mathbb{P}^n$ can be extended to a morphism $f : U \cup \{0\} \to \mathbb{P}^n$. On a non-empty open subset $U' \subset U$ we can write $f(x) = (f_0(x) : f_1(x) : \cdots : f_n(x))$ where $f_i \in k[x]$. Replace each f_i with $x^{-m} f_i$ where m is the largest power such that x^m divides f_i for all i. Then we can define $f : U' \cup \{0\} \to \mathbb{P}^n$ by $f(x) = (f_0(x) : \cdots : f_n(x))$. Thus our rational function is defined on $U \cup (U' \cup \{0\}) = U \cup \{0\}$ as required.

Problem 7. (a) If X has components $X_1, \dots, X_m$ then $\dim(X) = \max \dim(X_i)$.

(b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Solution: (b) We claim that if X and Y are both irreducible then so is $X \times Y$. Suppose $X \times Y = W_1 \cup W_2$ for closed subsets $W_i \subset X \times Y$ such that $W_2 \neq X \times Y$. If $(x, y) \notin W_2$ then $(\{x\} \times Y) \cap W_2 = \{x\} \times Z$ where $Z \subsetneq Y$ is a proper closed subset. Now for all $y' \in Y \setminus Z$ we must have $X \times \{y'\} \subset W_1$, so W_1 contains $X \times (Y \setminus Z)$. But this set is dense in $X \times Y$ so $W_1 = X \times Y$.

If X and Y are varieties and $X_0 \subsetneq X_1 \subsetneq \dots X_n \subset X$ and $Y_0 \subsetneq Y_1 \subsetneq \dots Y_m \subset Y$ are maximal chains of irreducible closed subsets, then $X_0 \times Y_0 \subsetneq X_0 \times Y_1 \subsetneq \cdots \subsetneq X_0 \times Y_m \subsetneq X_1 \times Y_m \subsetneq \cdots \subsetneq X_n \times Y_m$ is a chain in $X \times Y$ of length $m + n$. This shows that $\dim X \times Y \geq \dim X + \dim Y$.

For the opposite inequality we may assume that X and Y are both affine and irreducible. Choose $f_1, \dots, f_n \in k[X]$ such that $\{f_1, \dots, f_n\}$ is a transcendence basis for $k(X)/k$, and choose $g_1, \dots, g_m \in k[Y]$ similarly. Set $\tilde{f}_i = f_i \otimes 1$ and $\tilde{g}_j = 1 \otimes g_j \in k[X] \otimes_k k[Y] = k[X \times Y]$. Then $k(X \times Y)$ is algebraic over $k(\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g}_1, \dots, \tilde{g}_m)$, so $\text{tr.deg.}\, k(X \times Y) \leq n + m$.

Problem 8. The commutative algebra result *lying over* states that if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.

(a) Use lying over to show that if $\varphi : X \to Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of Y.

(b) If $\varphi : X \to Y$ is any morphism of varieties, then its image $\varphi(X)$ is *constructible*, i.e. a finite union of locally closed subsets of Y.

Solution: (a) WLOG X and Y are affine. Take $f_1, \dots, f_n \in k[X]$ such that $\{f_1, \dots, f_n\}$ is a trancendense basis for $k(X)/k(Y)$. Then the ring extensions $k[Y] \subset k[Y][f_1, \dots, f_n] \subset k[X]$ correspond to dominant morphisms $X \to Y \times \mathbb{A}^n \to Y$. Since the latter map is open it is enough to replace Y with $Y \times \mathbb{A}^n$, so we may assume that $k(X)$ is a finite extension of $k(Y)$.

Now let $k[X]$ be generated by $f_1, \dots, f_n$. Then each f_i is algebraic over $k(Y)$. This implies that each f_i is integral over $k[Y]_h$ for a suitable $h \in k[Y]$. Replacing X with X_h and Y with Y_h we may therefore assume that $k[Y] \subset k[X]$ is an integral extension. But in this case the morphism $X \to Y$ is surjective by "Lying over".

(b) Use induction on $\dim X$. It is enough to see that the image of each component of X is constructible, so WLOG X is irreducible. Since the image of $\varphi(X)$ is a constructible subset of Y if and only if it is a constructible subset of $\overline{\varphi(X)}$, we may furthermore assume that φ is dominant. But then $\varphi(X)$ contains a dense open subset $V \subset Y$ by part (a). Set $U = \varphi^{-1}(V) \subset X$. Since U is open in X, the set $W = X \smallsetminus U$ is a closed subvariety of X of dimension strictly smaller than the dimension of X. By induction this implies that $\varphi(W)$ is constructible, so the same is true for $\varphi(X) = \varphi(U) \cup \varphi(W) = V \cup \varphi(W)$.

Problem 10. Assume $\operatorname{char}(k) = 0$. Let $X = V_+(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \dots, x_n]$.

(a) Show that $X_{\text{sing}} = V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$.

(b) Show that $X_{\text{sing}} \neq X$.

Solution: (a) Apply the Jacobi criterion for affine varieties to one $D_+(x_i)$ at a time to obtain $X_{\text{sing}} = V_+(f, \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}) = V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$. The last equality holds in characteristic zero because $(\deg f) \cdot f = \sum_{i=0}^n x_i \frac{\partial f}{\partial x_i}$.

(b) WLOG X is irreducible, so f is an irreducible polynomial. If $V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$ $= V_+(f)$ then $\frac{\partial f}{\partial x_i} \in \sqrt{(f)}$ for each i. Therefore f must divide $(\frac{\partial f}{\partial x_i})^N$ for some N. Since f is irreducible, this implies that $(\frac{\partial f}{\partial x_i}) = 0$ for each i, a contradiction.

Problem 11. [Shafarevich II.1.13]

(a) Show that an intersection of r hypersurfaces in $\mathbb{P}^r$ is never empty.

(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that X contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq n/2$. Prove that X is singular. [Hint: Choose the coordinates on $\mathbb{P}^n$ such that $L = V_+(x_{r+1}, x_{r+2}, \dots, x_n) \subset \mathbb{P}^n$.]

Solution: (a) If $f_1, \dots, f_r \in k[x_0, \dots, x_r]$ are non-constant homogeneous polynomials then each component of $V(f_1, \dots, f_r) \subset \mathbb{A}^{r+1}$ has dimension at least one. And there is at least one such component because $0 \in V(f_1, \dots, f_r)$.

(b) WLOG $L = V_+(x_{r+1}, x_{r+2}, \dots, x_n)$. Since $L \subset V_+(f)$ it follows that $f \in (x_{r+1}, \dots, x_n)$. This implies that $\frac{\partial f}{\partial x_i} \in (x_{r+1}, \dots, x_n)$ for all $1 \leq i \leq r$, i.e.

$L \subset V(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_r})$. Since there are only $n - r \leq r$ equations $\frac{\partial f}{\partial x_i} = 0$ left, it follows from (a) that $L \cap V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}) \neq \emptyset$.

Problem 12. [Shafarevich II.1.10].

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If X has two different singular points, then X contains the line joining them.

Solution: WLOG $(1 : 0 : 0 : \cdots : 0)$ and $(0 : 1 : 0 : \cdots : 0)$ are singular points of X. Since $\frac{\partial f}{\partial x_i}(1, 0, 0, \dots, 0) = 0$ for all i, it follows that x_0^2 does not divide any of the monomials in f. Similarly x_1^2 does not divide any monomial in f. But this means that $f(s, t, 0, \dots, 0) = 0$ for all $s, t \in k$, so X contains the line $\{(s : t : 0 : \cdots : 0)\}$ through the given two singular points.

Problem 14. [Mostly Hartshorne I.6.3]

Give examples of varieties X and Y, a point $P \in X$, and a morphism $\varphi : X \smallsetminus \{P\} \to Y$ such that φ can't be extended to a morphism on all of X in each of the cases:

(a) X is a non-singular curve and Y is not projective.
(b) X is a curve, P is a singular point on X, Y is projective.
(c) X is non-singular of dimension at least two, Y is projective.

Solution: (a) $\mathrm{id} : (\mathbb{P}^1 - \{\infty\}) \longrightarrow \mathbb{A}^1$.
(b) $(V(y^2 - x^3 - x^2) - \{(0, 0)\}) \longrightarrow \mathbb{P}^1$; $(x, y) \mapsto (x : y)$.
(c) $(\mathbb{A}^{n+1} - \{0\}) \longrightarrow \mathbb{P}^n$.

Problem 17. Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$. Show that if $P \in E$ is any point then $E \smallsetminus \{P\}$ is affine.

Solution: Let $P = (x_0, y_0)$ and set $U = E - \{P\}$. If $\mathrm{char}(k) = 2$ or if $y_0 = 0$ then $U = D(x - x_0)$ is affine. If $\mathrm{char}(k) \neq 2$ and $y_0 \neq 0$ then

$$h = \frac{y + y_0}{x - x_0} = \frac{y^2 - y_0^2}{(x - x_0)(y - y_0)} = \frac{x^3 - x_0^3 - x + x_0}{(x - x_0)(y - y_0)} = \frac{x^2 + x x_0 + x_0^2 - 1}{y - y_0}$$

defines a regular function $h \in k[U]$ and $h \cdot (x - x_0) - (y - y_0) = 2y_0 \in k^*$. It follows the functions $x - x_0$ and $y - y_0$ generate the unit ideal in $k[U]$. Since the non-vanishing sets $U_{x-x_0} = E_{x-x_0}$ and $U_{y-y_0} = E_{y-y_0}$ are both affine (because E is affine), it follows that U is affine.

Problem 18. [Hartshorne I.6.2]

Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$, $\mathrm{char}(k) \neq 2$.

(a) E is a non-singular curve.
(b) The units in $k[E]$ are the non-zero elements of k. [Hints: Define an automorphism $\sigma : k[E] \to k[E]$ fixing x and sending y to $-y$. Then define a norm $N : k[E] \to k[x]$ by $N(a) = a\,\sigma(a)$. Show that $N(1) = 1$ and $N(ab) = N(a)N(b)$.]
(c) $k[E]$ is not a unique factorization domain.
(d) Show that E is not rational.

Solution: (a) Jacobi criterion.

(b) If $a, b \in k[E]$ satisfy $ab = 1$ then $N(a)N(b) = N(1) = 1$ so $N(a) \in k^*$. Write $a = f(x) + y\,g(x)$ where $f, g \in k[x]$. Then $N(a) = f^2 - (x^3 - x)g^2$. By looking at the highest term in this expression one sees that $N(a)$ can only be constant if $f \in k^*$ and $g = 0$, i.e. $a \in k^*$.

(c) $y^2 = x(x - 1)(x + 1)$.

(d) If E is rational then E is isomorphic to an open subset of $\mathbb{P}^1$. Since E is not projective, E is in fact an open subset of $\mathbb{A}^1$, which implies that $k[E]$ is a localization of $k[\mathbb{A}^1]$, thus a UFD.