ALGEBRAIC GEOMETRY I, PROBLEM SET 3 SOLUTIONS

Problem 1. Resolution of singularities for curves.

Let X be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve $\tilde{X}$ with a finite morphism $\varphi : \tilde{X} \to X$ such that the restriction $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a "proper" morhism φ.)

Solution: Resolution of singularities for curves.

Set $K = k(X)$ and let $\varphi : C_K \dashrightarrow X$ be the birational map given by the identity map on K. Let $U \subset C_K$ be the largest open set where φ is defined as a morphism. We must show that $\varphi(U) = X$. Let $V \subset X$ be any open affine subset. Then "finiteness of integral closure" implies that $B = \overline{k[V]}$ is a Dedekind domain, so $\tilde{V} = \text{Spec-m}\, B$ is a non-singular curve. The identity map on K now results in a commutative diagram of morphisms and rational maps

$$\begin{array}{ccc} C_K & \dashrightarrow & X \\ \uparrow & & \uparrow \\ \tilde{V} & \longrightarrow & V \end{array}$$

which shows that $\tilde{V} \subset U$. Since $\tilde{V} \to V$ is surjective by "lying over", this proves that $\varphi(U) = X$.

Problem 5. [Hartshorne I.5.3 and I.5.4]

Let $X \subset \mathbb{P}^2$ be a curve and $P \in \mathbb{P}^2$ any point. Let $I_{X,P} \subset \mathcal{O}_{\mathbb{P}^2,P}$ be the ideal of functions $f \in \mathcal{O}_{\mathbb{P}^2,P}$ such that $f|_{U \cap X} = 0$ for some open set U containing P. The multiplicity $\mu_P(X)$ of X at P is the largest number r such that $I_{X,P} \subset \mathfrak{m}_P^r$ where $\mathfrak{m}_P \subset \mathcal{O}_{\mathbb{P}^2,P}$ is the maximal ideal.

(a) $P \in X \Leftrightarrow \mu_P(X) \geq 1$.

(b) P is a non-singular point of X iff $\mu_P(X) = 1$.

(c) Let $Y \subset \mathbb{P}^2$ be another curve such that $X \cap Y$ is a finite set. Show that if $P \in X \cap Y$ then $I(X \cdot Y; P) = \dim_k \mathcal{O}_{\mathbb{P}^2,P}/(I_{X,P} + I_{Y,P})$.

(d) $I(X \cdot Y; P) = 1$ iff P is a non-singular point of both X and Y, and the tangent directions at P are different.

(e) $I(X \cdot Y; P) \geq \mu_P(X) \cdot \mu_P(Y)$.

(f) For all but a finite number of lines $L \subset \mathbb{P}^2$ through P we have $\mu_P(X) = I(X \cdot L; P)$.

Solution: (c) We may assume $P = (0\!:\!0\!:\!1) \in \mathbb{P}^2$. Set $S = k[x, y, z]$, and let $Q = I(P) = (x, y)$, $I(X) = (f)$, and $I(Y) = (g) \subset S$. Then $I(X \cdot Y; P) = \text{length}\, S_Q/(f, g)$. Set $R = \mathcal{O}_{\mathbb{P}^2,P} = k[\frac{x}{z}, \frac{y}{z}]_{(\frac{x}{z}, \frac{y}{z})}$ and $I = I_{X,P} + I_{Y,P} \subset R$. Then I is generated by $f' = f(\frac{x}{z}, \frac{y}{z}, 1)$ and $g' = g(\frac{x}{z}, \frac{y}{z}, 1)$. We must show that $\text{length}\, S_Q/(f, g) = \dim_k R/I$.

Now $S_Q = k[x,y,z]_{(x,y)} = k(z)[\frac{x}{z},\frac{y}{z}]_{(\frac{x}{z},\frac{y}{z})} = R \otimes_k k(z)$. Tensoring the exact sequence $R^{\oplus 2} \xrightarrow{(f',g')} R \to R/I \to 0$ with $k(z)$ we obtain $S_Q/(f,g) \cong R/I \otimes_k k(z)$. Thus length $S_Q/(f,g) = \dim_{k(z)} S_Q/(f,g) = \dim_k R/I$ as claimed.

(e) Let $X = V(f)$ and $Y = V(g) \subset \mathbb{A}^2$ and assume $P = (0,0)$. Set $m = \mu_P(X)$, $n = \mu_P(Y)$, $S = k[x,y]$, and $Q = I(P) = (x,y) \subset S$. Then $I(X \cdot Y; P) = \dim_k S_Q/(f,g) \geq \dim_k S_Q/(f,g,Q^{m+n}) = \dim_k S/(f,g,Q^{m+n})$. Now using the exact sequence

$$S/Q^n \times S/Q^m \xrightarrow{(f,g)} S/Q^{m+n} \to S/(f,g,Q^{m+n}) \to 0$$

we obtain $\dim_k S/(f,g,Q^{m+n}) \geq \binom{m+n+1}{2} - \binom{m+1}{2} - \binom{n+1}{2} = mn$ as required.

Problem 14. A morphism $f : X \to Y$ of varieties is called *affine* if for every open affine set $V \subset Y$ the inverse image $f^{-1}(V)$ is also affine. f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for all open affine $V \subset Y$.

Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall\ i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

Solution: Assume that $Y = V_1 \cup \cdots \cup V_m$ is an open affine covering such that $f^{-1}(V_i)$ is affine for each i. Let $U \subset Y$ be any open affine subvariety. We must show that $f^{-1}(U)$ is affine.

Given $h \in k[V_i]$, the non-vanishing set $(V_i)_h = \{y \in V_i \mid h(y) \neq 0\}$ is affine. Furthermore, $f^{-1}((V_i)_h) = f^{-1}(V_i)_{f^*h}$ is a non-vanishing set in the affine variety $f^{-1}(V_i)$, so $f^{-1}((V_i)_h)$ is affine. Since the sets $(V_i)_h$ form a basis for the topology of V_i, we may cover the intersection $V_i \cap U$ with such sets. It follows that U has an open covering $U = \bigcup V_i'$ such that V_i' and $f^{-1}(V_i')$ are affine for each i. By replacing Y with U and X with $f^{-1}(U)$, we may assume that $Y = U$ is affine. We must prove that X is affine.

Let $h \in k[Y]$ be a regular function such that $Y_h \subset V_i$ for some i. Then Y_h is affine and $f^{-1}(Y_h) = f^{-1}((V_i)_h) = f^{-1}(V_i)_{f^*h}$ is affine. Since the sets Y_h form a basis for the topology of Y, we may choose $h_1, \ldots, h_N \in k[Y]$ such that $Y = Y_{h_1} \cup \cdots \cup Y_{h_N}$ and $f^{-1}(Y_{h_i})$ is affine for each i. The first condition implies that $(h_1, \ldots, h_N)$ is the unit ideal in $k[Y]$. Set $g_i = f^*(h_i) \in k[X]$. Then $(g_1, \ldots, g_N)$ is the unit ideal in $k[X]$ and $X_{g_i} = f^{-1}(Y_{h_i})$ is affine for each i. It follows that X is affine.