

Derivations

R com. ring. A com. R -algebra. M A -module.

R -derivation $D: A \longrightarrow M$:

(1) R -linear.

(2) $D(ab) = a.D(b) + b.D(a)$, $a, b \in A$.

Note: If D satisfies (2), then (1) $\Leftrightarrow D(R) = 0$.

$\text{Der}_R(A, M) = \{ D: A \longrightarrow M \text{ } R\text{-derivation} \}$

A -module: $(b.D + D')(a) = b.D(a) + D'(a)$.

Example $A = k[x_1, \dots, x_n]$, $D: A \longrightarrow M$ any k -derivation.

$$D(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} D(x_i).$$

$\text{Der}_k(A, M) \cong M^{\oplus n}$ as A -module.

$\phi: A \longrightarrow B$ R -algebra hom, N B -module.

$$0 \longrightarrow \text{Der}_A(B, N) \longrightarrow \text{Der}_R(B, N) \xrightarrow{\phi_*} \text{Der}_R(A, N).$$

$$D \longmapsto D \circ \phi$$

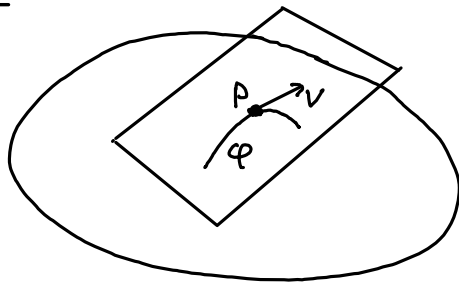
Tangent and cotangent vectors

X manifold, $p \in X$.

Tangent vector $v \in T_p X$:

Equiv. class of param. curves

$\varphi: \mathbb{R} \rightarrow X$ with $\varphi(0) = p$.



Given $C^\infty f: X \rightarrow \mathbb{R}$:

$$D_v(f) = \left. \frac{d}{dt} f(\varphi(t)) \right|_{t=0}.$$

$C^\infty(X)$ -module: $\mathbb{R}(p) = \mathbb{R}$, $f \cdot a = f(p)a$.

$$D_v \in \text{Der}_{\mathbb{R}}(C^\infty(X), \mathbb{R}(p)) =: T_p X.$$

$df_p \in (T_p X)^*$ cotangent vector: $df_p(v) = D_v(f)$.

Local ring of variety

X irred. variety, $p \in X$.

$$\mathcal{F} = \{ (U, f) \mid p \in U \subseteq X \text{ open, } f: U \rightarrow k \text{ regular} \}$$

$$\text{Equiv. rel: } (U, f) \sim (U', f') \Leftrightarrow f|_{U \cap U'} = f'|_{U \cap U'}$$

Local ring at p: $\mathcal{O}_{X,p} = \mathcal{F}/\sim = \{ f \in k(X) \mid f \text{ def. at } p \}$

$$\mathfrak{m}_p = \{ f \in \mathcal{O}_{X,p} \mid f(p) = 0 \} \subseteq \mathcal{O}_{X,p} \text{ unique max. ideal.}$$

$$k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \cong k \text{ is } \mathcal{O}_{X,p}\text{-module: } f \cdot a = f(p)a$$

Example: X affine, $p \in X$.

$I(p) \subseteq k[X]$ max. ideal.

$$\mathcal{O}_{X,p} = k[X]_{I(p)} = (k[X] - I(p))^{-1} k[X].$$

this def. is
valid when
 X not irred.

Zariski tangent space

$T_p X = \text{Der}_k(\mathcal{O}_{X,p}, k(p))$ tangent space.

$T_p^* X = \mathfrak{m}_p / \mathfrak{m}_p^2$. cotangent space.

Note: $D \in T_p X$, $f \in \mathfrak{m}_p^2 \Rightarrow D(f) = 0$.

$$g, h \in \mathfrak{m}_p \Rightarrow D(gh) = g(p) D(h) + h(p) D(g) = 0.$$

Note: $\text{Der}_k(\mathcal{O}_{X,p}, k(p)) \xrightarrow{\cong} (\mathfrak{m}_p / \mathfrak{m}_p^2)^*$

$$D \longmapsto [f + \mathfrak{m}_p^2 \mapsto D(f)]$$

$$[f \mapsto \overline{D(f - f(p) + \mathfrak{m}_p^2)}] \longleftarrow \overline{D}$$

$\therefore$ Perfect pairing $T_p^* X \times T_p X \longrightarrow k$
 $(f + \mathfrak{m}_p^2, D) \mapsto D(f)$

Def $p \in X$ is a non-sing. point if $\dim_k(T_p X) = \dim(X)$.

Jacobi Criterion

Exer: X affine, $p \in X$.

$$T_p X = \text{Der}_k(\mathcal{O}_{X,p}, k(p)) \xrightarrow{\cong} \text{Der}_k(\mathcal{O}(X), k(p))$$

Assume $X \subseteq \mathbb{A}^n$ closed, $p \in X$.

$$I = I(X) = \langle f_1, \dots, f_t \rangle \subseteq S = k[x_1, \dots, x_n].$$

$$T_p \mathbb{A}^n = \text{Der}_k(S, k(p)) = \text{Span}_k \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$$

$$\text{where } \frac{\partial}{\partial x_j}(f) = \frac{\partial f}{\partial x_j}(p).$$

$$T_p X = \{ D \in T_p(\mathbb{A}^n) \mid D(f_i) = 0 \text{ for } 1 \leq i \leq t \}$$

$$0 \longrightarrow T_p X \longrightarrow T_p \mathbb{A}^n \xrightarrow{J(p)} k^t$$

$$J = \left[\frac{\partial f_i}{\partial x_j} \right]_{t \times n} \quad \text{Jacobi matrix.}$$

$$\therefore \text{rank } J(p) = n - \dim T_p X \leq n - \dim \mathcal{O}_{X,p}$$

$$p \in X \text{ non-singular} \Leftrightarrow \text{rank } J(p) = n - \dim \mathcal{O}_{X,p}.$$