ALGEBRAIC GEOMETRY I, PROBLEM SET 1

Some problems are easy, others are trickier. You are encouraged to collaborate about the hard ones. Remember, serious people should do all these problems, ask for hints if required. Have fun!

Problem 1. Show that $I(\mathbb{A}^n) = (0)$.

Problem 2. If $I \subset R$ is any ideal, show that $\sqrt{I}$ is a radical ideal.

Problem 3. (a) $S \subset I(V(S))$.
(b) $W \subset V(I(W))$.
(c) If W is an algebraic set then $W = V(I(W))$.
(d) If $I \subset k[x_1, \dots, x_n]$ is any ideal then $V(I) = V(\sqrt{I})$ and $\sqrt{I} \subset I(V(I))$.

Problem 4. [Hartshorne I.1.2 and I.1.11]
(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$ is closed in $\mathbb{A}^3$ and find $I(X)$.
(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$ of $\mathbb{A}^3$.
(c) Show that $I(Y)$ can't be generated by less than three polynomials.
Hint: Is $I(Y)$ a graded ideal? Are you sure??

Problem 5. Let R be a commutative ring. The following are equivalent:
(a) R is Noetherian.
(b) Every ascending chain of ideals in R stabilizes.
(c) Every non-empty collection of ideals of R has a maximal element.

Problem 6. Show that $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible closed subset of $\mathbb{A}^3$ and find $I(W)$.
Hint: Construct a homomorphism $k[x, y, z] \to k[T]$ with kernel $I(W)$.

Problem 7. Find $\sqrt{(y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3)}$.

Problem 8. Let X be a Noetherian topological space.
(a) If an irreducible closed set Y is contained in a union $\bigcup X_i$ of finitely many closed sets X_i, then $Y \subset X_i$ for some i.
(b) X has finitely many components.
(c) X is the union of its components.
(d) X is not the union of any proper subset of its components.

Problem 9. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \to k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Problem 10. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \to k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.