

## ALGEBRAIC GEOMETRY I, PROBLEM SET 4

**Problem 1.** Let  $E$  be the elliptic curve  $V_+(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$  and let  $f, g : E \dashrightarrow \mathbb{P}^1$  be the rational maps defined by  $f(x : y : z) = (x : z)$  and  $g(x : y : z) = (y : z)$ . (These are just projections to the  $x$  and  $y$  axis on the open subset  $D_+(z)$ .)

- (a) Find the maximal open sets in  $E$  where  $f$  and  $g$  are defined as morphisms.
- (b) Find the degrees of the field extensions  $k(t) \subset k(E)$  induced by  $f$  and  $g$ .
- (c) Find the cardinality of  $f^{-1}(p)$  and  $g^{-1}(p)$  when  $p \in \mathbb{P}^1$  is a typical point. (Part of the exercise is to define what “typical” means.)

**Problem 2.** Let  $X$  be a projective variety and  $\varphi : \mathbb{P}^1 \dashrightarrow X$  any rational map. Show that  $\varphi$  is defined as a morphism on all of  $\mathbb{P}^1$ .

**Problem 3.** (a) If  $X$  has components  $X_1, \dots, X_m$  then  $\dim(X) = \max \dim(X_i)$ .  
 (b)  $\dim(X \times Y) = \dim(X) + \dim(Y)$ .

**Problem 4.** The commutative algebra result *lying over* states that, if  $R \subset S$  is an integral extension of commutative rings and  $P \subset R$  is a prime ideal, then there is some prime  $Q \subset S$  such that  $Q \cap R = P$ .

- (a) Use lying over to show that if  $\varphi : X \rightarrow Y$  is a dominant morphism of irreducible varieties, then  $\varphi(X)$  contains a dense open subset of  $Y$ .
- (b) If  $\varphi : X \rightarrow Y$  is any morphism of varieties, then its image  $\varphi(X)$  is *constructible*, i.e. a finite union of locally closed subsets of  $Y$ .

**Problem 5.** [Hartshorne I.5.2]

Assume  $\text{char}(k) \neq 2$ . Locate the singular points of the surfaces  $X = V(xy^2 - z^2)$ ,  $Y = V(x^2 + y^2 - z^2)$ , and  $Z = V(xy + x^3 + y^3)$  in  $\mathbb{A}^3$ . (Take a look at the nice pictures in Hartshorne!)

**Problem 6.** Assume  $\text{char}(k) = 0$ . Let  $X = V_+(f) \subset \mathbb{P}^n$  be a hypersurface given by a square-free homogeneous polynomial  $f \in k[x_0, \dots, x_n]$ .

- (a) Show that  $X_{\text{sing}} = V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$ .
- (b) Show that  $X_{\text{sing}} \neq X$ .

**Problem 7.** [Shafarevich II.1.13]

- (a) Show that an intersection of  $r$  hypersurfaces in  $\mathbb{P}^r$  is never empty.
- (b) Let  $X \subset \mathbb{P}^n$  be a hypersurface of degree at least two, such that  $X$  contains a linear subspace  $L \subset \mathbb{P}^n$  of dimension  $r \geq n/2$ . Prove that  $X$  is singular. [Hint: Choose the coordinates on  $\mathbb{P}^n$  such that  $L = V_+(x_{r+1}, x_{r+2}, \dots, x_n) \subset \mathbb{P}^n$ .]

**Problem 8.** [Shafarevich II.1.10].

Let  $X \subset \mathbb{P}^n$  be a hypersurface of degree three. If  $X$  has two different singular points, then  $X$  contains the line joining them.

**Problem 9.** If  $X$  is a variety and  $x \in X$ , we define the Zariski cotangent space to  $X$  at  $x$  to be  $\mathfrak{m}_x/\mathfrak{m}_x^2$ . The Zariski tangent space is the dual vector space  $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ . Show that if  $f : X \rightarrow Y$  is a morphism of varieties with  $f(x) = y$ , then  $f$  induces linear maps  $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$  and  $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \rightarrow (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$ .