GROTHENDIECK-RIEMANN-ROCH THEOREM

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1. The topic

This is a proposal for a first topic in Intersection Theory. The goal in the topic is to understand the Grothendieck-Riemann-Roch theorem and Prof. William Fulton's proof of it. The topic has been worked out under Prof. Madhav Nori's supervision. In doing the topic I have read F.A.C. [3], most of chapters 1-3 in Hartshorne's book [2], and chapters 1-8 plus chapter 15 in Fulton's book [1]. Furthermore I used Borel and Serre's article on Grothendieck-Riemann-Roch theorem [4], and chapter 5 in Altman and Kleiman's book on Grothendieck duality [5]. Of these, Fulton's book has been the main reference.

During the topic I have done a number of exercises in Hartshorne's book. Fulton's book does not contain exercises, however it has taken a lot of work to understand and verify most of the examples. I also plan to do exercises from J. Harris' book [6] to see more examples of algebraic varieties.

2. Intersection Theory

A very simple problem in Intersection Theory is the following: If $f(X) \in \mathbb{C}[X]$ is a nonzero polynomial of degree d, then how many solutions $a \in \mathbb{C}$ exist to the equation

$$f(a) = 0\,?$$

The answer is simple: If you count properly, then there are d solutions.

The above problem has a natural generalization to several variables. If $f_1, \dots, f_n \in \mathbb{C}[X_1, \dots, X_n]$ are polynomials of degrees $d_1, \dots, d_n$, then how many solutions $a = (a_1, \dots, a_n) \in \mathbb{A}^n = \mathbb{C}^n$ exist to the set of equations

$$f_i(a) = f_i(a_1, \dots, a_n) = 0$$

for $1 \le i \le n$? Given that the number of solutions is finite, the answer to this question is almost as simple as in the above case. If you include solutions in the enlargement $\mathbb{P}^n$ of $\mathbb{A}^n$ and furthermore count properly, then there are exactly $\prod_i d_i$. This is a special case of Bézout's Theorem.

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Intersection Theory is a branch of Algebraic Geometry, of which Bézout's Theorem is a particularly nice example. The basic question in Intersection Theory is what do you get when you intersect two subvarieties of an algebraic variety.

3. The group of cycle classes on a scheme

Let k be a field. In the following a scheme will mean a Noetherian scheme of finite type over k.

If X is a scheme, a cycle on X is an element of the free Abelian group generated by all subvarieties of X. The cycle of X is defined as $[X] = \sum \operatorname{ord}_V(X)[V]$, where the sum is over all irreducible components of X, and $\operatorname{ord}_V(X)$ is the length of the local ring of V in X. Note that if W is a closed subscheme of X, then $[W]$ may be considered as a cycle on X.

On the group of cycles on X, we define rational equivalence. Two cycles are rationally equivalent, if their difference lie in the subgroup generated by the cycles $[\operatorname{div}(f)]$ for all subvarieties V of X and rational functions $f \in k(V)$. The group $A(X)$ of cycle classes on X is defined to be the group of cycles modulo rational equivalence. If X is pure dimensional, $A(X)$ has a natural grading, where the degree of a subvariety is equal to its codimension in X.

Certain types of morphisms of schemes $f : X \to Y$ give rise to homomorphisms between $A(X)$ and $A(Y)$. If f is proper one may define a push-forward $f_* : A(X) \to A(Y)$. If V is a subvariety of X, we put $f_*[V] = \deg(V/W)[W]$, where $W = f(V)$ and $\deg(V/W)$ is nonzero only if $\dim(V) = \dim(W)$, in which case it is defined as $\deg(V/W) = [k(V) : k(W)]$.

If f is a flat morphism (of some relative dimension), or an l.c.i. morphism, or if Y is a non-singular variety (and X is pure-dimensional), one may define a pull-back homomorphism $f^* : A(Y) \to A(X)$. If f is flat, this is given by $f^*[V] = [f^{-1}(V)]$ for a subvariety $V \subset Y$.

4. The Chow ring of a non-singular variety

Let X be a non-singular variety with subvarieties V and W of codimensions c_1 and c_2. One may define an intersection product $V \cdot W$ in $A(X)$, which is the class of a cycle on $V \cap W$, of degree $c_1 + c_2$. This intersection product makes $A(X)$ into a graded ring with unit element $[X]$.

In very nice situations, $V \cdot W$ is merely $[V \cap W]$. For example this is true if W is a hypersurface, not containing V. In general the formula is more likely to hold, if V and W meet transversally at their points of intersection, and when all components of $V \cap W$ have the expected codimension $c_1 + c_2$.

The intersection product commutes with pull-back homomorphisms, so $A(-)$ is a contravariant functor from non-singular varieties to commutative rings.

Bézout's Theorem may be reformulated as $A(\mathbb{P}^n) = \mathbb{Z}[t]/(t^{n+1})$, where t^i is the class of a subspace of codimension i in $\mathbb{P}^n$. To see that this version implies the above statement, let $f_1, \dots, f_n \in k[X_0, \dots, X_n]$ be homogeneous polynomials of degrees $d_1, \dots, d_n$, and let S_i be the hypersurface $V(f_i)$. Assume that the intersection $W = S_1 \cap \cdots \cap S_n$ is a finite set of points. Then we have

$$\begin{aligned} [S_1] \cdot \cdots \cdot [S_n] &= [W] \\ &= \sum_{P \in W} \operatorname{ord}_P(W)[P] \,. \end{aligned}$$

On the other hand $[S_i] = d_i t$ in $A(\mathbb{P}^n)$, so the above product is also equal to $(\prod_i d_i) t^n$. As t^n is the class of any rational point in $\mathbb{P}^n$, we see that W contains $\prod_i d_i$ points, if we count properly.

Bézout's theorem also has applications to counting solutions in more complicated situations. For example it predicts that the number of lines intersecting four given lines in $\mathbb{P}^3$ is two or infinite.

5. Chern classes

If L is a line bundle on a non-singular variety X, we define the Chern class of L to be the class $c_1(L) = [D]$ in $A(X)$, where D is the divisor corresponding to L.

Now let E be a vector bundle of rank r on X with a filtration

$$E = E_r \supset E_{r-1} \supset \cdots \supset E_0 = 0 \,,$$

such that the quotients $L_i = E_i / E_{i-1}$ are line bundles. Then we define the Chern roots of E to be the classes $\alpha_1 = c_1(L_1), \dots, \alpha_r = c_1(L_r)$. We define the i'th Chern class $c_i(E)$ of E to be i'th symmetric polynomial in the α_j. With the notation $A(X)_{\mathbb{Q}} = A(X) \otimes \mathbb{Q}$, we define the classes in $A(X)_{\mathbb{Q}}$

$$\begin{aligned} \operatorname{ch}(E) &= \sum_j \exp(\alpha_j) \\ \operatorname{td}(E) &= \prod_j Q(\alpha_j) \,, \end{aligned}$$

where $Q(x) = x/(1 - e^{-x}) = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots$. Here $\operatorname{ch}(E)$ is called the Chern character of E, $\operatorname{td}(E)$ the Todd class.

If E does not have a filtration as above, the Chern classes of E may still be defined. The Chern character and Todd class of E can then be defined as a polynomials in the Chern classes. If $c_i = c_i(E)$ is the i'th Chern class, then

$$\begin{aligned}\mathrm{ch}(E) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots \\ \mathrm{td}(E) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots\end{aligned}$$

The Chern character satisfies $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$ for E and F vector bundles on X, and $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$ for $0 \to E' \to E \to E'' \to 0$ an exact sequence of vector bundles. By this we can define a ring homomorphism from the Grothendieck group of vector bundles on X,

$$\mathrm{ch} : K(X) \to A(X)_{\mathbb{Q}} .$$

6. Grothendieck-Riemann-Roch theorem

Let $f : X \to Y$ be a proper morphism of non-singular varieties. Then f gives rise to a homomorphism of Grothendieck groups $f_* : K(X) \to K(Y)$, defined by

$$f_*[E] = \sum_{i \geq 0} (-1)^i [R^i f_* E] .$$

f also gives rise to a morphism $f_* : A(X) \to A(Y)$ as defined above. Grothendieck-Riemann-Roch theorem states that for any vector bundle E on X we have in $A(Y)_{\mathbb{Q}}$

$$f_*(\mathrm{ch}(E) \cdot \mathrm{td}(T_X)) = \mathrm{ch}(f_*[E]) \cdot \mathrm{td}(T_Y) .$$

If X is complete, we may take $Y = \mathrm{Spec}(k)$ to be a point, and we use the notation $\int_X \alpha = f_*(\alpha) \in A(Y) = \mathbb{Z}$ for any $\alpha \in A(X)$. Furthermore $K(Y) = \mathbb{Z}$, $\mathrm{td}(T_Y) = 1$, and $[R^i f_* E] = \dim_k H^i(X, E)$ in $K(Y)$. It follows that $f_*[E] = \chi(X, E)$. In this case Grothendieck-Riemann-Roch implies Hirzebruch's formula

$$\chi(X, E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(T_X) .$$

7. Applications to non-singular curves

Let X be a complete non-singular curve, $g = \dim_k H^1(X, \mathcal{O}_X)$ the genus of X, and $K = c_1(\omega_X)$ a canonical divisor. For any divisor D on X we define $\deg(D) = \int_X D$ and $\ell(D) = \dim_k H^0(X, L(D))$. Note that $\ell(D) > 0$ if and only if D is equivalent to an effective divisor. It follows from Serre duality that $\chi(X, L(D)) = \ell(D) - \ell(K - D)$. Since $T_X = \omega_X^\vee$, we get $\mathrm{td}(T_X) = 1 - \frac{1}{2}K$, and so by Hirzebruch's formula

$$1 - g = \chi(X, \mathcal{O}_X) = \int_X \mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}(T_X) = -\frac{1}{2}\deg(K)\,.$$

In particular K has even degree. Applying Hirzebruch to $L(D)$, we get

$$\ell(D) - \ell(K - D) = \int_X \exp(D)(1 - \frac{1}{2}K) = \deg(D) + 1 - g\,.$$

This is known as Riemann-Roch theorem for curves.

Not that if $\deg(D) < 0$, D can't be equivalent to an effective divisor, and so $\ell(D) = 0$. It follows that if $\deg(D) > \deg(K) = 2g - 2$, we have $\ell(D) = \deg(D) + 1 - g$.

8. Applications to non-singular surfaces

Let X be a complete non-singular surface, and let $c_i = c_i(T_X)$. Then $\chi(X, \mathcal{O}_X) = \frac{1}{12}\int_X (c_1^2 + c_2)$. If E is a vector bundle of rank r on X with Chern classes $d_i = c_i(E)$, we get

$$\chi(X, E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(T_X) = \frac{1}{2}\int_X (d_1^2 - 2d_2 + d_1c_1) + r\chi(X, \mathcal{O}_X)\,.$$

In case $E = L(D)$ is a line bundle, this says

$$\chi(X, L(D)) = \frac{1}{2}\int_X (D \cdot D - D \cdot K) + \chi(X, \mathcal{O}_X)\,,$$

where $K = c_1(\omega_X) = -c_1$ is a canonical divisor.

If D is an effective Cartier divisor, we have a short exact sequence $0 \to L(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$. We get the following formula for the arithmetic genus of D:

$$p_a(D) = 1 - \chi(X, \mathcal{O}_X) + \chi(X, L(-D)) = \frac{1}{2}\int_X (D \cdot D + D \cdot K) + 1\,.$$

In the special case $X = \mathbb{P}^2$ we have $\omega_X = \mathcal{O}(-3)$, so $K = -3h$, where h is the class of a hyperplane. We get

$$\chi(\mathbb{P}^2, \mathcal{O}(n)) = \frac{1}{2}(n^2 + 3n) + 1 = \frac{1}{2}(n+1)(n+2)\,.$$

If C is a curve of degree n on $\mathbb{P}^2$, we get

$$p_a(C) = \frac{1}{2}(n^2 - 3n) + 1 = \frac{1}{2}(n-1)(n-2)\,.$$

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, we have $A(X) = \mathbb{Z}[s, t]/(s^2, t^2)$, where $s = [0 \times \mathbb{P}^1]$ and $t = [\mathbb{P}^1 \times 0]$. From $T_X = \mathrm{pr}_1^*(T_{\mathbb{P}^1}) \oplus \mathrm{pr}_2^*(T_{\mathbb{P}^1}) = L(2s) \oplus L(2t)$, we find $K = -2(s+t)$ and $\mathrm{td}(T_X) = \mathrm{td}(L(2s)) \cdot \mathrm{td}(L(2t)) = (1+s)(1+t) = 1 + s + t + st$, and so $\chi(X, \mathcal{O}_X) = \int_X \mathrm{td}(T_X) = 1$. We find

$$\chi(\mathbb{P}^1 \times \mathbb{P}^1, L(ms + nt)) = mn + m + n + 1 = (m+1)(n+1)\,.$$

If C is a curve on X of bidegree (m, n), we have $[C] = ms + nt$, and so

$$p_a(C) = \frac{1}{2} \int_X (C \cdot C + C \cdot K) + 1 = mn - m - n + 1 = (m-1)(n-1)\,.$$

References

[1] William Fulton. *Intersection Theory*
[2] Robin Hartshorne. *Algebraic Geometry*
[3] J-P. Serre. *Faisceaux algébriques cohérents*
[4] A. Borel, J-P. Serre. *Le théorème de Riemann-Roch (d'aprés Grothendieck)*
[5] A. B. Altman, S. L. Kleiman. *Introduction to Grothendieck duality theory*
[6] J. Harris. *Algebraic Geometry*