ALGEBRAIC GEOMETRY PROBLEMS

Problem 1. Show that $I(\mathbb{A}^n) = (0)$.

Problem 2. If $I \subset R$ is any ideal, show that $\sqrt{I}$ is a radical ideal.

Problem 3. (a) $S \subset I(V(S))$.
(b) $W \subset V(I(W))$.
(c) If W is an algebraic set then $W = V(I(W))$.
(d) If $I \subset k[x_1, \ldots, x_n]$ is any ideal then $V(I) = V(\sqrt{I})$ and $\sqrt{I} \subset I(V(I))$.

Problem 4. [Hartshorne I.1.2 and I.1.11]
(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$ is closed in $\mathbb{A}^3$ and find $I(X)$.
(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$ of $\mathbb{A}^3$.
(c) Show that $I(Y)$ can't be generated by less than three polynomials.
Hint: Is $I(Y)$ a graded ideal? Are you sure??

Problem 5. Let R be a commutative ring. The following are equivalent:
(a) R is Noetherian.
(b) Every ascending chain of ideals in R stabilizes.
(c) Every non-empty collection of ideals of R has a maximal element.

Problem 6. Show that $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible closed subset of $\mathbb{A}^3$ and find $I(W)$.
Hint: Construct a homomorphism $k[x, y, z] \to k[T]$ with kernel $I(W)$.

Problem 7. Find $\sqrt{(y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3)}$.

Problem 8. Let X be a Noetherian topological space.
(a) If an irreducible closed set Y is contained in a union $\bigcup X_i$ of finitely many closed sets X_i, then $Y \subset X_i$ for some i.
(b) X has finitely many components.
(c) X is the union of its components.
(d) X is not the union of any proper subset of its components.

Problem 9. Let X be any space with functions and $Y \subset \mathbb{A}^n$ an affine variety. Show that a function $f : X \to Y$ is a morphism if and only if each coordinate function $f_i : X \to k$ is regular for $1 \le i \le n$.

Problem 10. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \to k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Problem 11. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \to k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.

Problem 12. (a) $k[\mathbb{A}^n \smallsetminus \{0\}] = k[x_1, \dots, x_n]$ for $n \geq 2$.
(b) $\mathbb{A}^n \smallsetminus \{0\}$ is not an affine variety for $n \geq 2$.
(c) Every global regular function on $\mathbb{P}^n$ is constant, i.e. $k[\mathbb{P}^n] = k$.
(d) $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Problem 13. Let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Problem 14. Define the *homogenization* of a polynomial $f \in k[x_1, \dots, x_n]$ to be $f^* = x_0^{\deg(f)} f(x_1/x_0, \dots, x_n/x_0)$. Equivalently, if we write $f = f_0 + f_1 + \cdots + f_d$, with f_i a form of degree i and $f_d \neq 0$, then $f^* = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + f_d \in k[x_0, x_1, \dots, x_n]$.

Given any ideal $I \subset k[x_1, \dots, x_n]$, let $I^* \subset k[x_0, x_1, \dots, x_n]$ be the homogeneous ideal generated by $\{f^* \mid f \in I\}$.
(a) Find an example where $I = \langle h_1, \dots, h_m \rangle$ and $I^* \neq \langle h_1^*, \dots, h_m^* \rangle$.
(b) Let $X \subset \mathbb{A}^n$ be a closed subvariety. Identify $\mathbb{A}^n$ with $D_+(x_0) \subset \mathbb{P}^n$ and let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Show that $I(\overline{X}) = I(X)^* \subset k[x_0, \dots, x_n]$.

Problem 15. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Problem 16. Show that if R is a finitely generated reduced k-algebra then the space with functions Spec-m(R) is an affine variety.

Problem 17. Let X be any space with functions. A map $\varphi : \mathbb{P}^n \to X$ is a morphism if and only if $\varphi \circ \pi : \mathbb{A}^{n+1} \smallsetminus \{0\} \to X$ is a morphism.

Problem 18. Prove that the Segre map $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ gives an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a closed subvariety of $\mathbb{P}^N$, where $N = nm + n + m$.

Problem 19. Let $\varphi : X \to Y$ be a morphism of spaces with functions and suppose $Y = \bigcup V_i$ is an open covering such that each restriction $\varphi : \varphi^{-1}(V_i) \to V_i$ is an isomorphism. Then φ is an isomorphism.

Problem 20. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x, y, z]$ of degree 2, then $C \cong \mathbb{P}^1$.

Problem 21. Let X and Y be spaces with functions and let (P, π_X, π_Y) and (P', π'_X, π'_Y) be two products of X and Y. Show that there is a unique isomorphism $\varphi : P \xrightarrow{\sim} P'$ such that $\pi_X = \pi'_X \circ \varphi$ and $\pi_Y = \pi'_Y \circ \varphi$.

Problem 22. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 23. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y.
(a) Show that X is separated.
(b) Show that $\mathbb{P}^n$ has this property.

Problem 24. [Hartshorne II.2.16 and II.2.17]

Let X be any pre-variety and $f \in k[X]$ a regular function.

(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i.]

(b) $k[D(f)] = k[X]_f$.

(c) Let R be a k-algebra and let $f_1, \ldots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \ldots, f_r) = R$. If R_{f_i} is a finitely generated k-algebra for each i, then R is a finitely generated k-algebra.

(d) Suppose $f_1, \ldots, f_r \in k[X]$ satisfy $(f_1, \ldots, f_r) = k[X]$ and $D(f_i)$ is affine for each i. Then X is affine.

Problem 25. Let E be the elliptic curve $V_+(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x : y : z) = (x : z)$ and $g(x : y : z) = (y : z)$. (These are just projections to the x and y axis on the open subset $D_+(z)$.)

(a) Find the maximal open sets in E where f and g are defined as morphisms.

(b) Find the degrees of the field extensions $k(t) \subset k(E)$ induced by f and g.

(c) Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point. (Part of the exercise is to define what "typical" means.)

Problem 26. Let X be a projective variety and $\varphi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that φ is defined as a morphism on all of $\mathbb{P}^1$.

Problem 27. (a) If X has components $X_1, \ldots, X_m$ then $\dim(X) = \max \dim(X_i)$.

(b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 28. The commutative algebra result *lying over* states that, if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.

(a) Use lying over to show that if $\varphi : X \to Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of Y.

(b) If $\varphi : X \to Y$ is any morphism of varieties, then its image $\varphi(X)$ is *constructible*, i.e. a finite union of locally closed subsets of Y.

Problem 29. [Hartshorne I.5.2]

Assume $\mathrm{char}(k) \neq 2$. Locate the singular points of the surfaces $X = V(xy^2 - z^2)$, $Y = V(x^2 + y^2 - z^2)$, and $Z = V(xy + x^3 + y^3)$ in $\mathbb{A}^3$. (Take a look at the nice pictures in Hartshorne!)

Problem 30. Assume $\mathrm{char}(k) = 0$. Let $X = V_+(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \ldots, x_n]$.

(a) Show that $X_{\mathrm{sing}} = V_+(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$.

(b) Show that $X_{\mathrm{sing}} \neq X$.

Problem 31. [Shafarevich II.1.13]

(a) Show that an intersection of r hypersurfaces in $\mathbb{P}^r$ is never empty.

(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that X contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq n/2$. Prove that X is singular. [Hint: Choose the coordinates on $\mathbb{P}^n$ such that $L = V_+(x_{r+1}, x_{r+2}, \ldots, x_n) \subset \mathbb{P}^n$.]

Problem 32. [Shafarevich II.1.10].

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If X has two different singular points, then X contains the line joining them.

Problem 33. If X is a variety and $x \in X$, we define the Zariski cotangent space to X at x to be $\mathfrak{m}_x/\mathfrak{m}_x^2$. The Zariski tangent space is the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Show that if $f : X \to Y$ is a morphism of varieties with $f(x) = y$, then f induces linear maps $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ and $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$.

Problem 34. [Mostly Hartshorne I.6.3]

Give examples of varieties X and Y, a point $P \in X$, and a morphism $\varphi : X \smallsetminus \{P\} \to Y$ such that φ can't be extended to a morphism on all of X in each of the cases:

(a) X is a non-singular curve and Y is not projective.

(b) X is a curve, P is a singular point on X, Y is projective.

(c) X is non-singular of dimension at least two, Y is projective.

Problem 35. Let X and Y be curves and $\varphi : X \to Y$ a birational morphism.

(a) X_{sing} is a proper closed subset of X.

(b) $\varphi(X_{\text{sing}}) \subset Y_{\text{sing}}$.

(c) If $y \in Y$ is a non-singular point, then $\varphi^{-1}(y)$ contains at most one point.

Problem 36. Two non-singular projective curves are isomorphic if and only if they have the same function field.

Problem 37. Resolution of singularities for curves.

Let X be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve $\tilde{X}$ with a finite morphism $\varphi : \tilde{X} \to X$ such that the restriction $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a "proper" morhism φ.)

Problem 38. Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$. Show that if $P \in E$ is any point then $E \smallsetminus \{P\}$ is affine.

Problem 39. [Hartshorne I.6.2]

Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$, $\text{char}(k) \neq 2$.

(a) E is a non-singular curve.

(b) The units in $k[E]$ are the non-zero elements of k. [Hints: Define an automorphism $\sigma : k[E] \to k[E]$ fixing x and sending y to $-y$. Then define a norm $N : k[E] \to k[x]$ by $N(a) = a\,\sigma(a)$. Show that $N(1) = 1$ and $N(ab) = N(a)N(b)$.]

(c) $k[E]$ is not a unique factorization domain.

(d) Show that E is not rational.

Problem 40. Let $m_0, m_1, \dots, m_N \in k[x_0, \dots, x_n]$ be all the monomials of degree d. The *Veronese embedding* is the map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by

$$v_d(x_0 : \dots : x_n) = (m_0(x_0, \dots, x_n) : \dots : m_N(x_0, \dots, x_n)) .$$

(a) Show that v_d is an isomorphism of $\mathbb{P}^n$ with a closed subvariety in $\mathbb{P}^N$.

(b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $S = V_+(f)$ where $f \in k[x_0, \dots, x_n]$ is an irreducible form of degree d. Show that $S = {v_d}^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

Problem 41. Let L_1, L_2, and L_3 be lines in $\mathbb{P}^3$ such that none of them meet.

(a) There exists a unique quadric surface $S \subset \mathbb{P}^3$ containing L_1, L_2, and L_3. [Hint: Start by applying an automorphism of $\mathbb{P}^3$ to make the lines nice.]

(b) S is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting L_1, L_2, and L_3.

(c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet L_1, L_2, or L_3. Then the number of lines meeting L_1, L_2, L_3, and L_4 is equal to the number of points in $L_4 \cap S$, which is one, two, or infinitely many.

Problem 42. An *algebraic group* is a pre-variety G together with morphisms $m : G \times G \to G$ and $i : G \to G$, and an identity element $e \in G$, such that G is a group in the usual sense when m is used to define multiplication and i maps any element to its inverse element.

(a) Show that $\mathrm{GL}_n(k)$ is an algebraic group.

(b) Show that any algebraic group is separated.

(c) Show that $\mathbb{P}^1$ is not an algebraic group, i.e. it is not possible to find morphisms $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $i : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying the group axioms.

(d) Challenge: How about $\mathbb{P}^n$ for $n \geq 2$?

Problem 43. Let G be an irreducible algebraic group acting on a variety X, i.e. we have a morphism $G \times X \to X$ such that the axioms for a group action are satisfied.

(a) Show that each orbit in X is locally closed.

(b) Each orbit is a non-singular variety.

Problem 44. Let $\mathrm{GL}_n(k)$ act on $\mathrm{Gr}(d, n)$ by $g.V = \{g(x) \mid x \in V\}$. Show that for any points $V_1, V_2 \in \mathrm{Gr}(d, n)$ there exists an element $g \in \mathrm{GL}_n(k)$ such that $g.V_1$ and $g.V_2$ are both in $U_{\{1,\dots,d\}} \subset \mathrm{Gr}(d, n)$. Conclude that $\mathrm{Gr}(d, n)$ is separated.

Problem 45. (a) Let $0 < p < q < n$ be integers and $E = k^n$. Show that the set $\{(V, W) \in \mathrm{Gr}(p, E) \times \mathrm{Gr}(q, E) \mid V \subset W\}$ is closed in $\mathrm{Gr}(p, E) \times \mathrm{Gr}(q, E)$.

(b) Let $0 < d_1 < d_2 < \cdots < d_m < n$ be integers and let $\mathrm{F}\ell(d_1, \dots, d_m; E)$ be the set of flags of subspaces $V_1 \subset V_2 \subset \cdots \subset V_m \subset E$ such that $\dim V_i = d_i$. Give this set a structure of projective variety.

Problem 46. Set $E = k^n$, $X = \mathrm{Gr}(d, E)$, and let $F_1 \subset F_2 \subset \cdots \subset F_n = E$ be a flag of subspaces such that $\dim F_i = i$. Given a sequence of integers $a = (0 < a_1 < a_2 < \cdots < a_d \leq n)$, let $\Omega_a^\circ(F_\bullet)$ be the set of all $V \in X$ such that $\dim(V \cap F_p) = i$ whenever $a_i \leq p < a_{i+1}$, $0 \leq i \leq d$. (We set $a_0 = 0$ and $a_{d+1} = n + 1$.)

(a) Show that $\Omega_a^\circ(F_\bullet) \cong \mathbb{A}^m$ where $m = \sum a_i - \binom{d+1}{2}$.

(b) Show that the orbits for the action of the upper triangular matrices on X are the sets $\Omega_a^\circ(F_\bullet)$ for all sequences a where $F_i = \mathrm{span}\{e_1, \dots, e_i\}$.

(c) The *Schubert varieties* in X are the closures $\Omega_a(F_\bullet) = \overline{\Omega_a^\circ(F_\bullet)}$. Find a singular Schubert variety in some Grassmannian.

Problem 47. Let $X \subset \mathbb{P}^5$ be the subset of points $(x_0 : \cdots : x_5)$ such that the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}$$

has rank one. Show that X is a non-singular rational closed subvariety of $\mathbb{P}^5$, and find its dimension and degree.

Problem 48. [Mostly Hartshorne I.7.1] In this problem, just find the numbers and give an argument why they are correct that *could* be expanded into a proof.

(a) Find the degree of $v_d(\mathbb{P}^n)$ in $\mathbb{P}^N$ where v_d is the Veronese embedding.

(b) Find the degree of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$.

(c) Challenge: Find the degree of $\mathrm{Gr}(2,5)$ in $\mathbb{P}^9$.

Problem 49. [Hartshorne I.5.3 and I.5.4]

Let $X \subset \mathbb{P}^2$ be a curve and $P \in \mathbb{P}^2$ any point. Let $I_{X,P} \subset \mathcal{O}_{\mathbb{P}^2,P}$ be the ideal of functions $f \in \mathcal{O}_{\mathbb{P}^2,P}$ such that $f|_{U\cap X} = 0$ for some open set U containing P. The multiplicity $\mu_P(X)$ of X at P is the largest number r such that $I_{X,P} \subset \mathfrak{m}_P^r$ where $\mathfrak{m}_P \subset \mathcal{O}_{\mathbb{P}^2,P}$ is the maximal ideal.

(a) $P \in X \Leftrightarrow \mu_P(X) \geq 1$.

(b) P is a non-singular point of X iff $\mu_P(X) = 1$.

(c) Let $Y \subset \mathbb{P}^2$ be another curve such that $X \cap Y$ is a finite set. Show that if $P \in X \cap Y$ then $I(X \cdot Y; P) = \dim_k \mathcal{O}_{\mathbb{P}^2,P}/(I_{X,P} + I_{Y,P})$.

(d) $I(X \cdot Y; P) = 1$ iff P is a non-singular point of both X and Y, and the tangent directions at P are different.

(e) $I(X \cdot Y; P) \geq \mu_P(X) \cdot \mu_P(Y)$.

(f) For all but a finite number of lines $L \subset \mathbb{P}^2$ through P we have $\mu_P(X) = I(X \cdot L; P)$.

Problem 50. Let $\mathcal{F}$ be a sheaf on X and $p \in X$ a point. Prove the following from the definition of the stalk $\mathcal{F}_p$:

(a) Each element of $\mathcal{F}_p$ has the form s_p for some section $s \in \mathcal{F}(U)$, $p \in U$.

(b) Let $s \in \mathcal{F}(U)$, $p \in U$. Then $s_p = 0 \Leftrightarrow s|_V = 0$ for some $p \in V \subset U$.

(c) Let $s \in \mathcal{F}(U)$. Prove that $s = 0$ if and only if $s_p = 0 \ \forall\, p \in U$.

Problem 51. [Hartshorne II.1.2]

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $U = \bigcup V_i$ of U and sections $t_i \in \mathcal{F}(V_i)$ such that $\varphi_{V_i}(t_i) = s|_{V_i}$ for all i.

Problem 52. [Hartshorne II.1.14]

Let $\mathcal{F}$ be a sheaf on X and $s \in \mathcal{F}(X)$ a global section. Show that the set $\{p \in X \mid s_p \neq 0\}$ is a closed subset of X.

Problem 53. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on X. Show that $\ker(\varphi)_p = \ker(\varphi_p)$ and $\mathrm{Im}(\varphi)_p = \mathrm{Im}(\varphi_p)$ for all $p \in X$.

Problem 54. Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ a sheaf on Y. Show that $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}$ for all $p \in X$.

Problem 55. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Show that the map $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ constructed in class is bijective.

Problem 56. (a) Let X be an affine variety, M a $k[X]$-module, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Show that $\mathrm{Hom}_{k[X]}(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$.

(b) If X is affine and M and N are $k[X]$-modules then $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = (M \otimes_{k[X]} N)^{\sim}$.

(c) If $f : X \to Y$ is a morphism of varieties and $\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_X$-module.

Problem 57. (a) X is a ringed space, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Then the assignment $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines an $\mathcal{O}_X$-module. It is denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(b) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also invertible and that $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

Problem 58. Let X be a scheme of characteristic $p > 0$, $F : X \to X$ the Frobenius morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Show that $F^*\mathcal{L} \cong L^{\otimes p}$.

Problem 59. A morphism $f : X \to Y$ of pre-varieties is called *affine* if, for every open affine subset $V \subset Y$, the inverse image $f^{-1}(V)$ is also affine. The morphism f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for every open affine $V \subset Y$.

(a) Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall\ i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

(b) If f is affine and Y is separated, then X is separated.

Problem 60. (a) Let X be a complete variety and $f : X \to Y = \text{Spec-m}(k)$ the unique morphism to a point. Show that $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

(b) Find a projective variety X and a birational morphism $f : X \to Y$ such that $f_*\mathcal{O}_X$ is not locally free on Y.

Problem 61. (a) $Y \subset \mathbb{P}^n$ is a hypersurface of degree d with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$. Show that $\mathcal{I}_Y \cong \mathcal{O}(-d)$.

(b) Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding, $N = \binom{n+d}{n} - 1$. Show that $(v_d)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$.

Problem 62. Let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be any non-constant morphism. Then $\dim \varphi(\mathbb{P}^n) = n$. Furthermore, φ is the composition of a Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$, a projection $\mathbb{P}(k^N) - \mathbb{P}(L) \to \mathbb{P}(k^N/L)$ for some linear subspace $L \subset k^N$, and an inclusion of a linear subspace $\mathbb{P}(k^N/L) \subset \mathbb{P}^m$.

Problem 63. (a) Let $\varphi : X \to Y$ be an affine morphism of pre-varieties. Show that if Y is separated then so is X.

(b) X is an irreducible affine variety, $U \subset X$ an open affine subset, $\bar{U} \subset \bar{X}$ their normalizations, and $\pi : \bar{X} \to X$ the normalization map. Show that $\pi^{-1}(U) = \bar{U}$.

(c) If X is any irreducible variety then $\pi : \bar{X} \to X$ is a finite morphism. Conclude that $\bar{X}$ is separated.

Problem 64. (a) If Y is a normal variety and $f : Y \to X$ a dominant morphism, then there exists a unique morphism $\bar{f} : Y \to \bar{X}$ such that $f = \pi \circ \bar{f}$.

(b) Give a counter example to (a) when f is not dominant.

Problem 65. $X = V(xy - z^2) \subset \mathbb{A}^3$ is normal. [Hint: $k[X] = k[x, xt, xt^2] \subset k(x,t)$ where $t = z/x$.]

Problem 66. If X is any normal rational variety then $\mathrm{C}\ell(X)$ is a finitely generated Abelian group.

Problem 67. (a) Let $X \subset \mathbb{P}^2$ be a non-singular curve of degree 3 and $P \in X$ a point. Show that $\dim_k \Gamma(X, \mathcal{L}(n[P])) \geq n$ for all n.

(b) Any proper open subset of X is affine.

Problem 68. (a) Let $F, G, H \in k[x,y,z]$ be forms such that $V_+(G,H,z) = \emptyset$ in $\mathbb{P}^2$. Show that if $zF \in (G,H)$ then $F \in (G,H)$. [Hint: Use that $G_0 = G(x,y,0)$ and $H_0 = H(x,y,0)$ are relatively prime.]

(b) Let $C \subset \mathbb{P}^2$ be a curve, and set $\mathcal{O}_C(n) = \mathcal{O}_{\mathbb{P}^2}(n)|_C$. Then $\Gamma(C, \mathcal{O}_C(n)) = (k[x,y,z]/I(C))_n$ for all $n \geq 0$. [Hint: If $C = V_+(H) \subset D_+(y) \cup D_+(z)$ and if σ is a global section of $\mathcal{O}_C(n)$ then $\sigma/y^n = F(x,y,z)/y^m$ and $\sigma/z^n = A(x,y,z)/z^m$ for forms $F, A \in k[x,y,z]$ of degree $m \geq n$. Now use part (a).]

(c) Define the *arithmetic genus* of C to be $1 - P_C(0)$ where $P_C(m)$ is the Hilbert polynomial of $C \subset \mathbb{P}^2$. Show that $p_a = \frac{(d-1)(d-2)}{2}$ where d is the degree of C and that $\dim_k \Gamma(C, \mathcal{O}_C(n)) = nd + 1 - p_a$ for all large integers n.

Problem 69. (a) Let $C \subset \mathbb{P}^2$ be a non-singular curve and $Y \subset \mathbb{P}^2$ an irreducible curve different from C. Set $Y.C = \sum_P I(Y \cdot C; P)\, P \in \mathrm{Div}(C)$. Show that $\mathcal{L}([Y])|_C \cong \mathcal{L}(Y.C)$ on C.

(b) Let $L = V_+(f)$ and $M = V_+(g) \subset \mathbb{P}^2$ be lines (not equal to C) where $f, g \in k[x,y,z]$ are linear forms. Then the divisor of $f/g \in k(C)$ is $(f/g) = L.C - M.C$.

Problem 70. Let $E \subset \mathbb{P}^2$ be an elliptic curve and $P_0 \in E$ any point. Show that the map $E \to \mathrm{C}\ell^\circ(E)$ given by $P \mapsto P - P_0$ is bijective.

Problem 71. Let $D : S \to M$ be an R-derivation and $p(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ a polynomial. Then $D(p(a_1, \dots, a_n)) = \sum_{i=1}^n \frac{\partial p}{\partial x_i}(a_1, \dots, a_n) D(a_i)$ for all elements $a_1, \dots, a_n \in S$.

Problem 72. Let $E = V_+(zy^2 - x^3 + z^2 x) \subset \mathbb{P}^2$, $\mathrm{char}(k) \neq 2$. Show that $\Omega_E \cong \mathcal{O}_E$. [Hint: Compute the divisor of the section $d(x/z)$ of Ω_E.]

Problem 73. Let $\phi : X \to Y$ be a morphism of varieties and let $P \in X$ be a point such that $\phi^* : \mathcal{O}_{Y,\phi(P)} \to \mathcal{O}_{X,P}$ is an isomorphism. Then there exists an open subset $U \subset Y$ such that $\phi(P) \in U$ and $\phi : \phi^{-1}(U) \xrightarrow{\cong} U$ is an isomorphism.

Problem 74. Let X be a variety and $V \subset X$ any subset. Then V inherits a structure of space with functions from X. Assume that V is a variety with this structure. Show that V is locally closed in X.

Problem 75. Let $f : X \dashrightarrow Y$ be a rational map of algebraic varieties. Show that there exists a birational morphism $\pi : \tilde{X} \to X$ such that the rational map $f\pi$ extends to a morphism of varieties $f\pi : \tilde{X} \to Y$.

Problem 76. Set $E = V(y^2 - x^3 - 3) \subset \mathbb{C}^2$, $P = (1,2) \in E$, and $U = E \smallsetminus \{P\}$.

(a) Show that U is an open affine subvariety of E.

(b) Challenge: U is not of the form $D(f)$ for any regular function $f \in \mathcal{O}_E(E)$.

Problem 77. Set $E = k^{n+1}$ and recall that $\mathbb{P}^n = \{\ell \subset E \mid \ell \text{ is a line through the origin of } E\}$. Define $S = \{(\ell, v) \in \mathbb{P}^n \times E \mid v \in \ell\}$, and let $\rho : S \to \mathbb{P}^n$ be the projection.

(a) S is a subbundle of rank 1 of the trivial vector bundle $\mathbb{P}^n \times E$.

Define an $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{L}$ by $\Gamma(U, \mathcal{L}) = \{ \text{ morphisms } s : U \to L \mid \rho s = 1_U \}$.

(b) $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.

Let $\pi : E \smallsetminus \{0\} \to \mathbb{P}^n$ be the projection. For $d \in \mathbb{Z}$ we define an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^n}(d)$ by $\Gamma(U, \mathcal{O}(d)) = \{s \in \mathcal{O}_E(\pi^{-1}(U)) \mid s(\lambda v) = \lambda^d s(v)\ \forall \lambda \in k, v \in E\}$.

(c) The sheaf $\mathcal{O}(d)$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.

(d) Find an integer $d \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(d)$ as an $\mathcal{O}_{\mathbb{P}^n}$-module.