Note that no books, notes, or calculators may used during the exam.

You will be given a table of the Laplace transform, based on Table III in our text. Some unneeded formulas will be omitted, and the formula from Theorem 4.4.3 of the text, for the Laplace transform of a periodic function, will be added. This table will be posted on the class web page several days before the exam, so that you can see just what you will get (but don't bother to bring this to the exam; you will be given a fresh copy.) Note that, like Table III, the supplied table will contain several useful general formulas, as well as the Laplace transforms of specific functions.

1. Find $\mathcal{L}\{te^{-2t}\}$ directly from the definition of the Laplace transform.

2. Find $\mathcal{L}\{te^{-t}\cos(2t)\}$ from formula 52 of Table III, then from formula 56. (You may use other formulas from the table as well).

3. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+2)}\right\}$ using formula 57 of Table III. Check your answer using partial fractions. Check your answer using formula 28 of the table.

4. Solve for $y(t)$ using the Laplace transform:

(a) $y'' + 25y = e^{-3t}$, $y(0) = 1$, $y'(0) = -2$; (b) $y'' - 4y' + 4y = \cos t$, $y(0) = 3$, $y'(0) = 0$.

5. In this problem we study the initial value problem

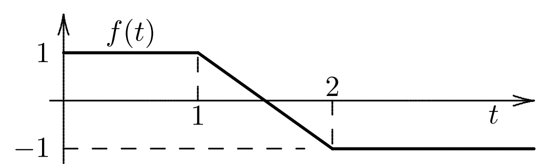
$$y'' + (\pi^2/4)y = \delta(t-3), \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Use the Laplace transform to find the solution $y(t)$.

(b) Evaluate explicitly $y(2)$, $y(3)$, and $y(4)$.

(c) Show that $y'(t)$ is discontinuous at $t = 3$.

6. (a) Let $f(t)$ be the function shown in the figure to the right ($f(t)$ is defined for $t \geq 0$). Use the unit step function $\mathcal{U}(t)$ to give a single formula for $f(t)$.



(b) Compute the Laplace transform of $f(t)$, using your answer in (a).

(c) Use the Laplace transform to solve $y' + y = f(t)$, $y(0) = 2$.

7. Use Laplace transforms to solve the Volterra integral equation

$$f(t) = t^2 + \int_0^t e^{-\tau} f(t-\tau)\, d\tau.$$

8. Find $\mathcal{L}\{\sin 3t\}$ from Theorem 4.3.3. Check your answer from Table III.

9. Let A be a given 15×15 matrix.

(a) Explain why the equations $A\mathbf{x} = \mathbf{0}$ must have a solution (a very easy answer!)

(b) Suppose that there is a vector $\mathbf{b}$ such that the system of equations $A\mathbf{x} = \mathbf{b}$ has no solution. What does this fact tell you about rank(A)?

(c) Assuming that the situation is as in (b), explain carefully why the system $A\mathbf{x} = \mathbf{0}$ must have a nontrivial solution. Your answer should involve a discussion of the possible REF or RREF of A.

(d) Suppose now that rank$(A) = 6$, and that the vector $\mathbf{b}$ is such that $A\mathbf{x} = \mathbf{b}$ has a solution. How many free parameters will the general solution to $A\mathbf{x} = \mathbf{b}$ contain?

10. Do the "extra" problems 1 and 2 from Assignment 4.

11. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & k \end{pmatrix}$, where k is a real number.

(a) Use Gaussian elimination (elementary row operations) to transform A into a matrix R in row-echelon form. Label each row operation you use by $cR_i + R_j$, R_{ij}, or cR_i as in the text, with specific values for the scalar c and the indices i, j.

(b) Use the result of **(a)** to determine the rank of A (your answer will depend on the value of k).

(c) Use the result of **(a)** to find all values of k for which the system

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 3 \\ 2x_1 & + & 3x_2 & = & 4 \\ 3x_1 & + & 5x_2 & = & k \end{array}$$

is consistent. Don't solve the system.

12. Let $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & -1 \\ -2 & 2 & 2 \end{pmatrix}$. Use Gaussian elimination to reduce A to a matrix R in row echelon form. Keep track of the row operations you use.

(b) Find $\det R$ and, *from this and your work in (a)*, find $\det A$. Explain your reasoning.

(c) Check your answer to (b) by a cofactor expansion along the second column of A.

(d) Compute A^{-1} from $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$. Check your work using the method $(A \mid I) \to (I \mid A^{-1})$.

13. Let $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A.

14. Let $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.

(a) What does the fact that A is symmetric tell you about its eigenvalues?

(b) Find the eigenvalues of A. Hint: they are all small integers.

(c) Find a diagonal matrix D and an orthogonal matrix P such that $D = P^T AP$.

15. Let A be a 4×4 real symmetric matrix with $\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$, and suppose that $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$, and $\mathbf{K}_4$ are nonzero vectors such that $A\mathbf{K}_i = \lambda_i \mathbf{K}_i$ for $i = 1, 2, 3, 4$. Classify each statement below as **T**rue or **F**alse:

- λ_1, λ_2, λ_3, and λ_4 must all be distinct.
- λ_1, λ_2, λ_3, and λ_4 must all be real.
- $\{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4\}$ must be an orthogonal set.
- If λ_1, λ_2, λ_3, and λ_4 are all distinct, then $\{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4\}$ must be an orthogonal set.
- If λ_1, λ_2, λ_3, and λ_4 are all distinct, then $\{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4\}$ must be an orthonormal set.
- One can find four nonzero vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{X}_4$ with $A\mathbf{X}_i = \lambda_i \mathbf{X}_i$ for $i = 1, \ldots, 4$ such that $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ is an orthonormal set.

Short answers. These have been checked fairly carefully, but if you don't get the given answer, be suspicious.

1. As for examples on Z&W page 198.
2. $(s^2+2s-3)/(s^2+2s+5)^2$.
3. $(1-e^{-2t})/2$.
4. (a) $(e^{-3t}+33\cos 5t-13\sin 5t)/34$. (b) $(3\cos t-4\sin t+72e^{2t}-140te^{2t})/25$.
5. (a) $y=\cos(\pi t/2)+2\mathcal{U}(t-3)\cos(\pi t/2)/\pi$; (b) $-1, 0, 1+2/\pi$.
6. (a) $1+(2-2t)\mathcal{U}(t-1)+(2t-4)\mathcal{U}(t-2)$; (b) $1/s-2e^{-s}/s^2+2e^{-2s}/s^2$;
   (c) $1+e^{-t}-2\mathcal{U}(t-1)(t-2+e^{1-t})+2\mathcal{U}(t-2)(t-3+e^{2-t})$.
7. $t^2+t^3/3$.
8. See formula 7 of Table III.
9. (a) Consider $\mathbf{x}=0$; (b) $\operatorname{rank}(A)<15$; (c) See Class Notes; (c) 9.
10. See short answers on Assignment 4.
11. (b) 3 if $k\neq 7$; 2 if $k=7$. (b) Only $k=7$.
12. (b,c) $\det A=32$; (d) $A^{-1}=\dfrac{1}{16}\begin{pmatrix}5 & 1 & -7\\ -1 & 3 & 3\\ 6 & -2 & -2\end{pmatrix}$.
13. $\lambda_1=4, \lambda_2=1, \mathbf{K}_1=\begin{pmatrix}-1\\ 2\end{pmatrix}, \mathbf{K}_2=\begin{pmatrix}1\\ 1\end{pmatrix}$.
14. (a) Real; (c) $D=\begin{pmatrix}6 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}$, $P=\begin{pmatrix}1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2}\\ -2/\sqrt{6} & 1/\sqrt{3} & 0\\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}\end{pmatrix}$.
15. F, T, F, T, F, T.