Note that no books, notes, or calculators may used during the exam.

You will be given a sheet of formulas related to trigonometric series. This sheet will be posted on the class web page several days before the exam, so that you can see just what you will get (but don't bother to bring this to the exam; you will be given a fresh copy.)

1. Let $f(x) = 1 - x$ for $0 \le x \le 3$.

(a) (i) Find the half-range sine series of $f(x)$ on $[0, 3]$.
(ii) This series is the Fourier series of a function $f_{\text{odd}}(x)$ defined for all x; sketch $f_{\text{odd}}(x)$ on $[-9, 9]$.
(iii) If $S_N^{(\text{odd})}(x)$ is the sum of the first N terms of the series, find $\lim_{N\to\infty} S_N^{(\text{odd})}(x)$ for $x = 0, 2, 3$.

(b) (i) Find the half-range cosine series of $f(x)$ on $[0, 3]$.
(ii) This series is the Fourier series of a function $f_{\text{even}}(x)$ defined for all x; sketch $f_{\text{even}}(x)$ on $[-9, 9]$.
(iii) If $S_N^{(\text{even})}(x)$ is the sum of the first N terms of the series, find $\lim_{N\to\infty} S_N^{(\text{even})}(x)$ for $x = 0, 2, 3$.

(c) (i) Find the Fourier series of $f(x)$ on $[0, 3]$.
(ii) This series is the Fourier series of a function $f_{\text{per}}(x)$ defined for all x; sketch $f_{\text{per}}(x)$ on $[-9, 9]$.
(iii) If $S_N^{(\text{per})}(x)$ is the sum of the first N terms of the series, find $\lim_{N\to\infty} S_N^{(\text{per})}(x)$ for $x = 0, 2, 3$.

2. Let $f(x) = e^{3x}$ for $-\pi < x < \pi$. Find the coefficients c_n (where $n = 0, \pm1, \pm2, \ldots$) in the complex Fourier series for $f(x)$. Simplify your answer using the relation $e^{\pi i} = -1$.

3. Consider the boundary value problem

$$5\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \pi, \quad t > 0; \tag{PDE}$$

$$u_x(0, t) = u_x(\pi, t) = 0, \qquad t > 0; \tag{BC}$$

$$u(x, 0) = x, \quad 0 < x < \pi. \tag{IC}$$

(a) Find all product solutions $u(x, t) = X(x)T(t)$ of (PDE) and (BC). Show all steps.

(b) Find the solution $u(x, t)$ of the full problem.

(c) Find $\lim_{t\to\infty} u(x, t)$. Explain why your answer is physically reasonable.

4. Consider the boundary value problem

$$9\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 2, \quad t > 0; \tag{PDE}$$

$$u(0, t) = u(2, t) = 0, \qquad t > 0; \tag{BC}$$

$$u(x, 0) = 3\sin\frac{\pi x}{2} - \sin\pi x, \quad u_t(x, 0) = 2\sin 6\pi x, \qquad 0 < x < 2. \tag{IC}$$

(a) Find all product solutions $u(x, t) = X(x)T(t)$ of (PDE) and (BC). Show all steps.

(b) Find the solution $u(x, t)$ of the full problem. Hint: very little calculation is required.

5. In this problem we work with the inner product $(f, g) = \int_{-1}^{1} f(x)g(x)\,dx$ on $[-1, 1]$.

(a) Show that $\phi_1(x) = 2$ and $\phi_2(x) = x$ are orthogonal on $[-1, 1]$ in this inner product, and then find constants A and B such that the function $\phi_3(x) = x^2 + Ax + B$ is orthogonal to both $\phi_1(x)$ and $\phi_2(x)$.

(b) Suppose that the function $f(x)$ is given by $f(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ for certain constants c_i, where $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ are as in (a) (with the A and B you found there). Give formulas for c_1, c_2, and c_3; each formula should involve one unevaluated integral.

6. In this problem we work with the inner product $(f,g) = \int_a^b f(x)g(x)\,dx$ on the interval $[a,b]$. Suppose that $\{\phi_1(x),\ldots,\phi_n(x)\}$ is an orthogonal set of functions and that $f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$. **Derive** the fact that $c_k = (f,\phi_x)/\|\phi_k\|^2$, $k = 1,\ldots,n$.

7. Solve Laplace's equation in the rectangle $0 < x < 2$, $0 < y < 3$, subject to the boundary conditions in (a) and (b) (two separate problems):

(*a*) $u_y(x,0) = 0,\quad u_y(x,3) = 0,\quad 0 < x < 2,\qquad u(0,y) = 2,\quad u(2,y) = \cos\pi y,\quad 0 < y < 3.$
(*b*) $u(x,0) = 2,\quad u(x,3) = 0,\quad 0 < x < 2,\qquad u(0,y) = 0,\quad u(2,y) = \sin 2\pi y/3,\quad 0 < y < 3.$

8. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < 4; \qquad y(0) = 0, \quad y(4) - 4y'(4) = 0.$$

(a) Show that 0 is an eigenvalue for this problem (we will write $\lambda_0 = 0$) and give the corresponding eigenfunction $y_0(x)$.

(b) Given that the remaining eigenvalues $\lambda_1 < \lambda_2 < \cdots$ are all positive, find an equation which each eigenvalue must satisfy and find the corresponding eigenfunction $y_n(x)$.

(c) Give a graphical construction of the eigenvalues, and from this find a good approximation for λ_n when n is very large. Label your graph clearly: axes, curves, etc.

(d) A continuous function $f(x)$ defined for $0 \le x \le 4$ will have an expansion $f(x) = \sum_{n=0}^{\infty} c_n y_n(x)$. Give the formula for c_n in terms of explicit integrals (involving $f(x)$ and possibly trigonometric functions and polynomials).

9. Suppose the function $x(t)$ has the Fourier cosine series $x(t) = \sum_{n=1}^{\infty} a_n \cos(nt)$ for $0 < t < \pi$.

(a) Find the Fourier cosine series for the function $x''(t) + 3x(t)$ on $0 < t < \pi$. Express your answer in terms of the coefficients a_n (no integration is needed).

(b) Suppose $f(t) = \displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^3+5}\cos(nt)$ for $0 < t < \pi$. Determine the coefficients a_n so that $x(t)$ will satisfy the differential equation $x''(t) + 3x(t) = f(t)$ for $0 < t < \pi$.

Short answers. I have tried to check these fairly carefully, but if you can't get the given answer, it might be my error.

1. (a) $f(x) \sim (2/\pi)\sum_{n=1}^{\infty}[(1+2(-1)^n)/n]\sin(n\pi x/3)$; $0, -1, 0$;
(b) $f(x) \sim -1/2 + (6/\pi^2)\sum_{n=1}^{\infty}[(1-(-1)^n)/n^2]\cos(n\pi x/3)$; $1, -1, -2$;
(c) $f(x) \sim -1/2 + (3/\pi)\sum_{n=1}^{\infty}(1/n)\sin(2n\pi x/3)$; $-1/2, -1, -1/2$.

2. $c_n = (-1)^n \sinh(3\pi)/(3\pi - in\pi)$.

3. (a) $u_0(x,t) = 1$, $u_n(x,t) = \cos(nx)e^{-5n^2 t}$. (b) $\pi/2 - (4/\pi)\sum_{n\text{ odd}} u_n(x,t)/n^2$. (c) $\pi/2$.

4. (a) $\sin(n\pi x/2)\cos(3n\pi t/2)$, $\sin(n\pi x/2)\sin(3n\pi t/2)$, $n = 1, 2, \ldots$;
(b) $3\sin(\pi x/2)\cos(3\pi t/2) - \sin(\pi x)\cos(3\pi t) + (1/9\pi)\sin(6\pi x)\sin(18\pi t)$

5. (a) $A = 0$, $B = -1/3$.
(b) $c_1 = (1/4)\int_{-1}^{1} f(x)\,dx$, $c_2 = (3/2)\int_{-1}^{1} xf(x)\,dx$, $c_3 = (45/8)\int_{-1}^{1} f(x)(x^2 - 1/3)\,dx$.

6. See text or class notes.

7. (a) $u(x,y) = (2-x) + \cos(\pi y)\sinh(\pi x)/\sinh(2\pi)$;
(b) $u(x,y) = \sin(2\pi y/3)\sinh(2\pi x/3)/\sinh(4\pi/3)$
$+\sum_{n\text{ odd}}(8/n\pi)[\cosh(n\pi y/2) - \coth(3n\pi/2)\sinh(n\pi y/2)]\sin(n\pi x/2)$.

8. (a) $y_0 = x$; (b) $\tan 4\sqrt{\lambda} = 4\sqrt{\lambda}$; $y(x) = \sin(\sqrt{\lambda}\,x)$; (c) $\lambda_n \sim [(2n+1)\pi/8]^2$;
(d) $c_0 = (3/64)\int_0^4 xf(x)\,dx$, $c_n = \int_0^4 f(x)\sin(\sqrt{\lambda_n}\,x)\,dx/\int_0^4 \sin^2(\sqrt{\lambda_n}\,x)\,dx$.

9. (a) $x''(t) + 3x(t) = \sum_{n=1}^{\infty}(3-n^2)a_n\cos nt$, (b) $a_n = (n+2)/[(n^3+5)(3-n^2)]$.