NOTES ON THE GAMMA AND ZETA FUNCTIONS

LEV A. BORISOV

ABSTRACT. These are notes on Γ and ζ functions provided for the students in my section of Math 403 at Rutgers in Spring 2014.

1. EULER'S Γ FUNCTION

The idea behind the Gamma function is the following. We know how to define $n! = 1 \cdot 2 \cdot \ldots \cdot n$ for a nonnegative integer n. Can we define some kind of $z!$ for a real, or perhaps even complex number z? It turns out that this is possible, and the solution is given by the Euler's Gamma function $\Gamma(z)$.

We first define $\Gamma(z)$ for $\Re(z) \geq 1$. [1]

Definition 1.1. We define

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} \, dt$$

for z with $\Re(z) \geq 1$.

It is nice to claim to define something, but there are some issues that need to be addressed regarding this definition before it starts to make sense.

First of all, we need to explain what we mean by t^{z-1}, because in general one can not raise arbitrary numbers to arbitrary powers unambiguously. Fortunately, t is a positive real number, so we can use

$$t^{z-1} = \mathrm{e}^{(z-1)\ln t}$$

where $\ln t$ is the usual logarithm of positive number t.

Second, more serious, issue is the convergence of the above integral. The size of the integrand in Definition 1.1 is

$$|t^{z-1}\mathrm{e}^{-t}| = |\mathrm{e}^{-t+(z-1)\ln t}| = \mathrm{e}^{-t+(\Re(z)-1)\ln t}$$

This is smaller than $\mathrm{e}^{-\frac{t}{2}}$ for t large enough, so for any z the integral is dominated at $+\infty$ by the absolutely convergent $\int^{+\infty} \mathrm{e}^{-\frac{t}{2}} \, dt$.

[1] $\Re(z)$ means the real part of z.

We also need to see what happens near 0. For $\Re(z) \geq 1$ we have for $t < 1$ the size of the integrand is bounded by 1. As a result, we can define a Riemann integral of this function on $[0, 1]$ since it is bounded and continuous almost everywhere. [2]

Our first order of business is to verify that $\Gamma(z)$ in Definition 1.1 is related to the factorial.

Proposition 1.2. *For $n \geq 1$ we have $\Gamma(n) = (n-1)!$.*

Proof. We will prove this by induction on n. For $n = 1$ we have

$$\Gamma(1) = \int_{t=0}^{+\infty} \mathrm{e}^{-t}\, dt = \lim_{B\to+\infty} \int_0^B \mathrm{e}^{-t}\, dt = \lim_{B\to+\infty} (1 - \mathrm{e}^{-B}) = 1 = 0!$$

For any z with $\Re(z) > 1$ we can integrate by parts

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\, dt = \lim_{B\to+\infty} \int_{1/B}^B t^{z-1}\mathrm{e}^{-t}\, dt$$

$$= \lim_{B\to+\infty} \left(\left(\frac{t^z}{z}\right)\mathrm{e}^{-t}\Big|_{1/B}^B - \int_{1/B}^B \left(\frac{t^z}{z}\right)(\mathrm{e}^{-t})'\, dt \right)$$

$$= 0 + \lim_{B\to+\infty} \int_{1/B}^B \left(\frac{t^z}{z}\right)\mathrm{e}^{-t}\, dt = 0 + \frac{1}{z}\Gamma(z+1) = \frac{\Gamma(z+1)}{z}.$$

Thus if we know that $\Gamma(k) = (k-1)!$, then

$$\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k!$$

which proves the induction step from $n = k$ to $n = k + 1$. $\square$

Remark 1.3. As far as I can tell, the reason behind t^{z-1} as opposed to t^z in the definition $\Gamma(z)$ is purely historical. Nonetheless, this is the universally accepted convention, which then leads to $\Gamma(n) = (n-1)!$.

Remark 1.4. The relation

$$\Gamma(z+1) = z\Gamma(z) \tag{1.1}$$

that we derived in the proof above is a key equation for a Γ function. It will be useful later.

Proposition 1.5. *The function $\Gamma(z)$ is complex differentiable in the half-plane $\Re(z) > 1$.*

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[2] If $\Re(z) > 1$, then the function can be extended to a continuous one by setting the value at 0 to be 0. In truth, the integral converges for $\Re(z) > 0$, but we will not need it.

Proof. This is a simple consequence of the standard statements about differentiating integrals that depend on a parameter. Specifically, we need to know that that the integral for the derivative is absolutely convergent, uniformly in some neighborhood of a given z. The derivative in question is

$$-e^{-t-(z-1)\ln t}\ln t.$$

It is then easy to estimate at both ends of $[0,+\infty]$. □

Remark 1.6. We are now interested in trying to analytically extend Γ to the rest of the complex numbers. We still want to have the property $\Gamma(z+1) = z\Gamma(z)$. Indeed, if it holds in the domain $\Re(z) > 1$, then it should continue to hold elsewhere. This shows that we can not extend Γ to $z = 0$, since $0\Gamma(0) = \Gamma(1) = 1$ is impossible. We similarly can not define Γ at negative integers.

Definition 1.7. For arbitrary complex number $z \notin \mathbb{Z}_{\leq 0}$ we define

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

for some n such that $\Re(z+n) > 1$.

Remark 1.8. We observe that Definition 1.7 makes sense, i.e. the result $\Gamma(z)$ does not depend on the choice of n. Indeed, from the equation (1.1) we have

$$\frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} = \frac{(z+n)\Gamma(z+n)}{z(z+1)\cdots(z+n)} = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

so we can replace n by $n+1$ in the above definition. So any sufficiently large n will give the same result.

Remark 1.9. It is clear from Definition 1.7 that $\Gamma(z)$ have simple poles of order 1 at $\mathbb{Z}_{\leq 0}$.

A remarkable property of the Gamma function is captured by the so-called reflection formula. I will present the proof of it which uses several nice ideas that may be useful in other situations.

Theorem 1.10. *For any* $z \notin \mathbb{Z}$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Proof. We observe that the left hand side is anti-periodic with period 1, just as the right-hand side. Indeed,

$$\Gamma(z+1)\Gamma(1-(z+1)) = \Gamma(z+1)\Gamma(-z) = \Gamma(z)z\Gamma(-z) = -\Gamma(z)\Gamma(1-z).$$

Thus if we can prove the statement for $0 < \Re(z) < 1$, then we can prove it for all $\Re(z) \notin \mathbb{Z}$, which then extends to $z \notin \mathbb{Z}$ by continuity.

We have

$$z(1-z)\Gamma(z)\Gamma(1-z) = \Gamma(z+1)\Gamma(2-z)$$
$$= \Big(\int_0^{+\infty} t^z \mathrm{e}^{-t}\,dt\Big)\Big(\int_0^{+\infty} s^{1-z}\mathrm{e}^{-s}\,ds\Big) = \iint_{[0,+\infty]^2} t^z s^{1-z}\mathrm{e}^{-s-t}\,dtds.$$

We will now switch the variables [3] to $u = s+t$ and $v = t$. The new region is $u \in [0,+\infty],\ 0 \le v \le u$. The Jacobian is 1, so $dtds = dvdu$. We get (after making another change of variables)

$$z(1-z)\Gamma(z)\Gamma(1-z) = \int_{u=0}^{+\infty}\int_{v=0}^{u} v^z(u-v)^{1-z}e^{-u}\,dvdu$$
$$= \int_{u=0}^{+\infty} e^{-u}\int_{w=0}^{1}(wu)^z(u-wu)^{1-z}\,d(wu)du$$
$$= \int_{u=0}^{+\infty}\int_{w=0}^{1} u^2e^{-u}w^z(1-w)^{1-z}\,dwdu$$
$$= \Big(\int_{u=0}^{+\infty} u^2\mathrm{e}^{-u}\,du\Big)\Big(\int_{w=0}^{1} w^z(1-w)^{1-z}\,dw\Big) = 2\int_{w=0}^{1} w^z(1-w)^{1-z}\,dw.$$

Note that we have used $(uw)^z = u^z w^z$ which we can because u and w are positive real numbers. We have definitely made some progress, since we reduced the problem to a single integral. The assumption $0 < \Re(z) < 1$ assures that the integral is OK at the ends of the segment.

We will now make one more change of variables $x = w/(1-w)$ $(w = x/(x+1))$ to put this integral in a familiar form. We have $0 \le x \le +\infty$ and $dx = \frac{1}{(1-w)^2}\,dw$. We then have

$$\frac{1}{2}z(1-z)\Gamma(z)\Gamma(1-z) = \int_{x=0}^{+\infty} x^z(1-w)^3\,dx = \int_{x=0}^{+\infty}\frac{x^z}{(x+1)^3}\,dx. \tag{1.2}$$

We are almost done. It remains to do a contour trick that we did once in Section 2.6 of the textbook. Specifically, consider the keyhole contour. Then the x^z analytically continues to $x^z\mathrm{e}^{2\pi\mathrm{i}z}$ on the lower side of the contour, and we have

$$(1-\mathrm{e}^{2\pi\mathrm{i}z})\int_{x=0}^{+\infty}\frac{x^z}{(x+1)^3}\,dx = 2\pi\mathrm{i}\mathrm{Res}_{x=-1}\frac{x^z}{(x+1)^3} = 2\pi\mathrm{i}\frac{1}{2}(x^z)''|_{x=-1}$$

[3] Some care needs to be taken when switching variables in improper integrals, but the function goes to zero fast enough to not cause trouble. The motivation behind this change of variables is to simplify e^{-s-t} to e^{-u}. Still, it would be useless, if not for the next change of variables.

$$= \pi \mathrm{i} z(z-1)(-1)^z = \pi \mathrm{i} z(z-1)\mathrm{e}^{\pi \mathrm{i} z}.$$

Putting this together with (1.2), we get

$$\Gamma(z)\Gamma(z-1) = -\frac{2\pi \mathrm{i} \mathrm{e}^{\pi \mathrm{i} z}}{1-\mathrm{e}^{2\pi \mathrm{i} z}} = -\pi \frac{2\mathrm{i}}{\mathrm{e}^{-\pi \mathrm{i} z} - \mathrm{e}^{\pi \mathrm{i} z}} = \frac{\pi}{\sin(\pi z)}.$$

□

Corollary 1.11. *The Gamma function* $\Gamma(z)$ *is never* 0*.*

Remark 1.12. As a consequence of the reflection formula, we observe that

$$\Gamma(\frac{1}{2})^2 = \frac{\pi}{\sin \frac{\pi}{2}} = \pi.$$

It is easy to see that $\Gamma(\frac{1}{2}) = 2\Gamma(\frac{3}{2}) > 0$, so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Remark 1.13. These notes are just a very first intro to the Gamma function. It has many more beautiful properties.

Homework problems.

1. Verify from Definition 1.7 that $z\Gamma(z) = \Gamma(z+1)$.

2. Calculate the residues of $\Gamma(z)$ at its poles. Hint: use the reflection formula.

3. Calculate the values of $\Gamma(z)$ at $z = -\frac{3}{2}$ and $z = \frac{5}{2}$.

4. Prove $\overline{\Gamma(z)} = \Gamma(\bar{z})$.

5. Find an explicit formula for $|\Gamma(1+\mathrm{i}y)|$.

2. Riemann ζ function.

This section of the notes introduces you to the Riemann's zeta function. The ultimate goal is to explain the statement of the famous Riemann hypothesis.

We first define $\zeta(s)$ for complex numbers s with $\Re(s) > 1$ by means of the convergent series.

Definition 2.1. For any s in the half-plane $\Re(s) > 1$ define

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Remark 2.2. The meaning of n^s is $\mathrm{e}^{s \ln n}$, thus $|n^s| = |n^{\Re(s)}|$ and the series is absolutely convergent by the p-series test.

Remark 2.3. Values of ζ at positive *even* integers are reasonably well-understood. For example

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

More generally, $\zeta(2k) = c_k \pi^{2k}$ where c_k is some rational number related to the so-called Bernoulli number. The contour integration argument that we used in one of the homework problems to calculate $\zeta(2)$ works for any other positive even integer.

Remark 2.4. In contrast, values of ζ at positive *odd* integers larger than one are much harder to work with. It is a difficult result of Apéry, published in 1979, that $\zeta(3)$ is irrational. The expectation is that $\zeta(k)$ for odd k are mutually transcendental, but current methods are not even close to verifying this.

Proposition 2.5. *Riemann zeta function $\zeta(s)$ is analytic in $\Re(s) > 1$.*

Proof. It follows from general statements about differentiating a series that depends on parameter. One can switch differentiation and summation as long as one has (locally) uniform convergence of the sum of the derivatives. In our case this is

$$\sum_{n=1}^{\infty} (-n^{-s} \ln n)$$

whose absolute values are not hard to bound by $\frac{1}{n^{\frac{1}{2}(\Re s+1)}}$. ☐

The importance of the ζ function is that its behavior is intimately linked with the behavior of the set of prime numbers. For example, we can rewrite it as an infinite product over the set of prime numbers p.

Proposition 2.6. *For $\Re(s) > 1$ there holds*

$$\zeta(s) = \prod_{p=\text{prime}} \frac{1}{1 - p^{-s}}.$$

Proof. Consider the partial product on the right hand side for $p \leq N$. Expand each term of the product as a geometric series in p^{-s} for $p \leq N$ to get

$$(1 + 2^{-s} + 2^{-2s} + ...)(1 + 3^{-s} + 3^{-2s} + ...)(1 + 5^{-s} + 5^{-2s} + ...)...$$

which expands to $\sum n^{-s}$ over all n which have prime factors at most N. [4] In the limit $N \to +\infty$ we get the sum $\sum n^{-s}$ over all n, which is

[4] We crucially use the fact the every integer is uniquely written as a product of powers of primes.

equal to $\zeta(s)$. The absolute convergence of the series for $\zeta(s)$ assures that one can open brackets in the product of the above series. $\square$

Given an analytic function which is defined on an open set, it is always natural to ask whether this function can be extended to analytic function on a larger set, hopefully on almost all of $\mathbb{C}$. If such extension, called analytic continuation exists, then it is unique. In what follows, we will work on extending ζ.

Proposition 2.7. *For $\Re(s) > 1$ we have*

$$\zeta(s) = \frac{s}{s-1} - s\int_{x=1}^{+\infty} \{x\}x^{-s-1}\,dx$$

where $\{x\}$ is the fractional part of x.

Proof. We have

$$\int_{x=1}^{+\infty} \{x\}x^{-s-1}\,dx = \int_{x=1}^{+\infty} x^{-s}\,dx - \int_{x=1}^{+\infty} [x]x^{-s-1}\,dx$$

where $[x]$ is the floor function and the convergence is by the p-test. Then

$$\int_{x=1}^{+\infty} \{x\}x^{-s-1}\,dx = \frac{1}{s-1} - \sum_{n=1}^{+\infty}\int_{x=n}^{n+1} [x]x^{-s-1}\,dx$$

$$= \frac{1}{s-1} - \sum_{n=1}^{+\infty}\int_{x=n}^{n+1} nx^{-s-1}\,dx = \frac{1}{s-1} + \sum_{n=1}^{+\infty} \frac{nx^{-s}}{s}\Big|_{n}^{n+1}$$

$$= \frac{1}{s-1} + \frac{1}{s}\sum_{n=1}^{+\infty}(n(n+1)^{-s} - n^{1-s})$$

$$= \frac{1}{s-1} + \frac{1}{s}(\sum_{n=1}^{+\infty}((n+1)^{1-s} - n^{1-s}) - (n+1)^{-s})$$

$$= \frac{1}{s-1} - \frac{1}{s}\sum_{n=1}^{+\infty}(n+1)^{-s} + \frac{1}{s}\sum_{n=1}^{+\infty}((n+1)^{1-s} - n^{1-s})$$

$$= \frac{1}{s-1} - \frac{1}{s}(\zeta(s)-1) + \frac{1}{s}(\lim_{n\to+\infty}((n+1)^{1-s} - 1^{1-s})$$

$$= \frac{1}{s-1} - \frac{1}{s}\zeta(s).$$

Here we have used the telescoping series trick for $\sum_{n=1}^{+\infty}((n+1)^{1-s} - n^{1-s})$. The result now follows. $\square$

We can use the above integral presentation of ζ to extend it to a differentiable function on $\Re(s) > 0, s \neq 1$.

Proposition 2.8. *The integral* $\int_{x=1}^{+\infty} \{x\} x^{-s-1}\, dx$ *converges absolutely for* $\Re(s) > 0$ *to an analytic function.*

Proof. We have $|\{x\}| < 1$ and $|x^{-s-1}| = |x^{-\Re(s)-1}|$, so the convergence follows by a comparison to a standard convergent integral. Analyticity is proved similarly to Proposition 2.5. $\square$

Corollary 2.9. *The function* $\zeta(s)$ *extends to an analytic function on* $\Re(s) > 0, s \neq 1$ *with a pole of order one at* $s = 1$.

Proof. Combine Proposition 2.7 and Proposition 2.8. $\square$

We can now formulate the famous Riemann Hypothesis. We define the critical strip to be the set of s with $0 < \Re(s) < 1$.

Conjecture 2.10. (Riemann Hypothesis.) If $\zeta(s) = 0$ some s in the critical strip, then $\Re(s) = \frac{1}{2}$. In other words, zeros of ζ in the critical strip occur only on the line $\Re(s) = \frac{1}{2}$.

Fortunately, Riemann Hypothesis has never been as popular among non-mathematicians as the Fermat's Last Theorem, presumably because it is harder to state. However, among mathematicians it is certainly considered to be one of the top if not the most famous unsolved problem. Please do not try to prove Riemann Hypothesis. You are not in any position to do so and will only annoy the professionals.

I want to also mention without proof the remarkable symmetry of ζ function which allows one to extend it to the whole $\mathbb{C}$.

Proposition 2.11. *Consider the function* $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. *Then* $\xi(s)$ *satisfies for* $0 < \Re(s) < 1$

$$\xi(1-s) = \xi(s)$$

Proof. Too hard for this course. $\square$

Corollary 2.12. *Function* $\zeta(s)$ *can be extended to a function on all of* $\mathbb{C}$ *with pole only at* $s = 1$.

Proof. Proposition 2.11 ensures that $\xi(s)$ can be extended to $\mathbb{C}$ with isolated poles by declaring $\xi(s) := \xi(1-s)$ for $\Re(s) \leq 0$.

It remains to look at the order of the poles. We know that in $\Re(s) > 0$ the only pole is $s = 1$. For $\Re(s) \leq 0$ we have

$$\zeta(s) = \xi(s)\pi^{\frac{s}{2}}\Gamma(\frac{s}{2})^{-1} = \xi(1-s)\pi^{\frac{s}{2}}\Gamma(\frac{s}{2})^{-1}$$

$$= \zeta(1-s)\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\pi^{\frac{s}{2}}\Gamma(\frac{s}{2})^{-1} = \zeta(1-s)\pi^{\frac{2s-1}{2}}\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2})^{-1}.$$

Because Γ has poles at nonpositive integers and no zeros, we see that for $\Re(s) \leq 0$ the only possible pole comes from $\zeta(1-s)$ term. This is a pole of order one when $1-s=1$, i.e. $s=0$. However, then $\Gamma(\frac{s}{2})^{-1}$ has a zero which cancels the pole. □

Homework problems.

1. Prove that for $\Re(s) > 1$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}.$$

Here $\mu(n)$ is 0 if n is divisible by a square. If n is not divisible by a square and is a product of k distinct primes, then $\mu(n) = (-1)^k$.

2. Prove that $\zeta(s) \neq 0$ for $\Re(s) > 1$.

3. Prove that $\zeta(-2k) = 0$ for all positive integer k and that these are the only zeroes of $\zeta(s)$ for $\Re(s) < 0$.

4. Prove that for $\Re(s) > 1$ there holds

$$\Gamma(s)\zeta(s) = \int_0^{+\infty} \frac{x^{s-1}}{\mathrm{e}^x - 1}\,dx.$$

Hint. Expand $\frac{1}{\mathrm{e}^x-1}$ as a (geometric) power series in e^{-x} and then use substitution of $t = nx$ to get to the integral for Γ.

5. Show that $\zeta(-1) = -\frac{1}{12}$. This is sometimes expressed as (tongue-in-cheek) identity

$$1 + 2 + 3 + \cdots = -\frac{1}{12}.$$

Mathematics Department, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08540, USA

E-mail address: borisov@math.rutgers.edu