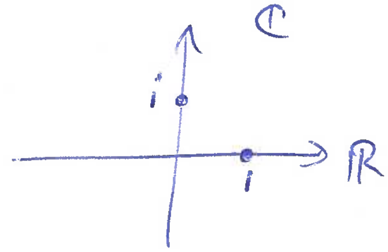
# Complex Variables 1/16/2018 FIRST CLASS

Complex plane: $\mathbb{C} = \mathbb{R}^2$

complex number: $z = (x, y) \in \mathbb{R}^2$




Identify $\mathbb{R}$ with $x$-axis: $x = (x, 0)$

Notation: $1 = (1, 0)$, $i = (0, 1)$

$$z = (x, y) = x \cdot 1 + y \cdot i = x + iy$$

Operations: $\boxed{\text{Set } i^2 = -1}$

$$z = x + iy = (x, y)$$
$$w = s + it = (s, t)$$

$$z + w = (x+s) + i(y+t) = (x+s, y+t)$$
$$zw = (x+iy)(s+it) = xs + iys + xit + i^2yt$$
$$= (xs - yt) + i(ys + xt)$$

$$\boxed{(x,y)\cdot(s,t) = (xs - yt, xt + ys)}$$

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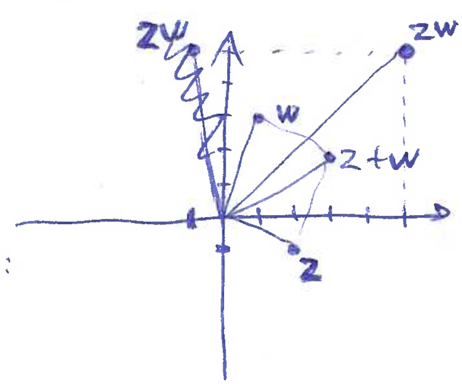
Example $z = 2 - i$, $w = 1 + 3i$

$$zw = (2-i)(1+3i) = (2+3) + i(-1+6) = 5 + 5i$$
$$z + w = 3 + 2i$$




Check:

For $a, b, c \in \mathbb{C}$: All usual identities hold:

$$a + b = b + a$$
$$(a+b) + c = a + (b+c)$$
$$ab = ba$$
$$(ab)c = a(bc)$$
$$a(b+c) = ab + ac$$

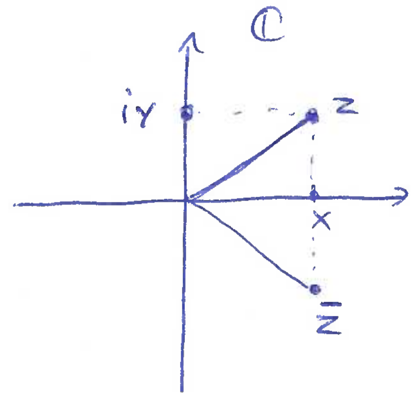
Notation

$$z = (x,y) = x + iy \in \mathbb{C}$$

$x = \mathrm{Re}(z)$ real part

$y = \mathrm{Im}(z)$ imaginary part

$|z| = \sqrt{x^2 + y^2}$ modulus / abs. value




Complex conjugate: $\bar{z} = x - iy$

OBS

$$z + \bar{z} = 2x = 2\,\mathrm{Re}(z).$$

$$z - \bar{z} = 2y = 2\,\mathrm{Im}(z)$$

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 + iyx - xiy = |z|^2$$

Inverse of $z$: $z \neq 0 \Rightarrow z^{-1} = \frac{1}{|z|^2}\bar{z}$

$$z \cdot z^{-1} = \frac{1}{|z|^2} z\bar{z} = 1.$$

$\therefore$ $\mathbb{C}$ is a field: Every nonzero elt has mult inverse.

Example $z = 2 - i$, $w = 1 + 3i$

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{2+i}{4+1} = \frac{2}{5} + i\frac{1}{5}$$

$$\frac{w}{z} = \frac{\bar{z}w}{z\bar{z}} = \frac{(2+i)(1+3i)}{5} = \frac{2-3 + i(1+6)}{5}$$

$$= -\frac{1}{5} + \frac{7}{5}i$$

## Polar coordinates

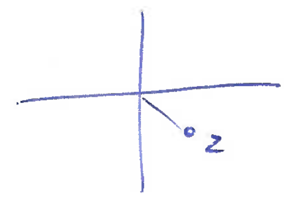
$z = (x,y) = x + iy$

write $(x,y) = (r\cos\theta,\ r\sin\theta)$.

$r = |z|$.

$\therefore\quad z = |z|(\cos\theta,\ \sin\theta) = |z|(\cos\theta + i\sin\theta)$

$z = |z|\cos(\theta) + i|z|\sin(\theta)$




Example $z = 1 - i$

$|z| = \sqrt{2} \qquad \theta = -\frac{\pi}{4}$

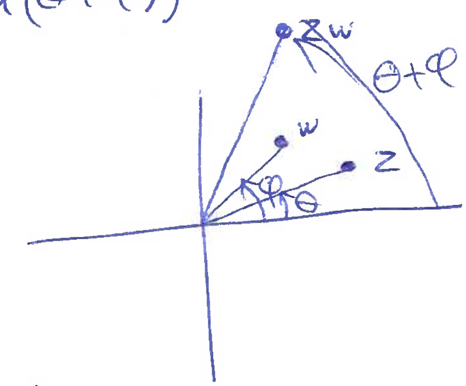
$1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

Multiplication: $z = |z|(\cos\theta + i\sin\theta) \quad w = |w|(\cos\varphi + i\sin\varphi)$

$$z \cdot w = |z|(\cos\theta + i\sin\theta)\cdot|w|(\cos\varphi + i\sin\varphi)$$

$$= |z||w|\Big(\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\Big)$$

$$= |z||w|\left(\cos(\theta+\varphi) + i\sin(\theta+\varphi)\right)$$




~~Inverse:~~ $\cancel{\bar z = |z|(\cos(\theta) + i\sin(-\theta))}$

$\cancel{z^{-1} = \frac{\bar z}{|z|^2} = \frac{1}{|z|^2}|z|(\cos}$

Conjugate: $\bar z = |z|(\cos\theta - i\sin\theta)$

$= |z|(\cos(-\theta) + i\sin(-\theta))$

Inverse $\quad \dfrac{1}{z} = \dfrac{\bar z}{|z|^2} = |z|^{-1}\left(\cos(-\theta) + i\sin(-\theta)\right)$

**Division:**

$$\frac{w}{z} = z^{-1}w = \frac{|w|}{|z|}\left(\cos(\varphi-\theta) + i\sin(\varphi-\theta)\right).$$

**Notation:** If $z = r(\cos\theta + i\sin\theta)$, $\theta \in \mathbb{R}$

then $\theta$ is an argument of $z$

write: $\arg(z) = \theta$. Means $z = |z|(\cos\theta + i\sin\theta)$

**Note** $\arg(z)$ is NOT unique.

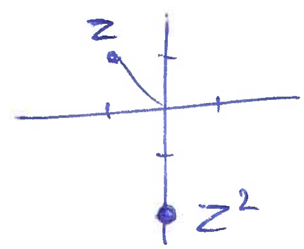
$\theta + n\cdot(2\pi)$ also works, any $n \in \mathbb{Z}$.

**Def** For $z \in \mathbb{C}$, $z \neq 0$, Let $\mathrm{Arg}(z)$ be the unique number $\theta_0 \in [-\pi, \pi)$ such that

$$z = |z|(\cos\theta_0 + i\sin\theta_0)$$

We have: $\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w) \pmod{2\pi}$.

**Example** $z = -1 + i$




$$\mathrm{Arg}(z) = \frac{3\pi}{4}$$

$$z^2 = (-1+i)(-1+i) = -2i$$

$$\mathrm{Arg}(z^2) = -\frac{\pi}{2}$$

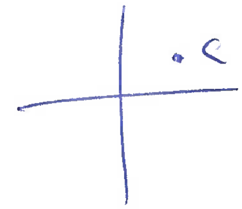
$$= 2\,\mathrm{Arg}(z) - 2\pi.$$

**Other issue:** $\mathrm{Arg}: \mathbb{C}\setminus\{0\} \to [-\pi, \pi)$ is NOT continuous.

$$\mathrm{Arg}: \mathbb{C}\setminus\{0\} \longrightarrow [-\pi, \pi)$$

## Mandelbrot Set

$$\boxed{z' = z^2 + c}$$




$c \in \mathbb{C}$.

$f_c(z) = z^2 + c$

Consider $(0,\ f_c(0),\ f_c(f_c(0)),\ f_c f_c f_c(0), \ldots)$

$f_c^n(0) = \underbrace{f_c f_c \cdots f_c}\,(0)$

what happens as $n \to \infty$ ??

$c = -2$: $f_{-2}(0) = -2$. $f_{-2}(-2) = 2$. $f_2(2) = 2$

$(0, -2, 2, 2, 2, \ldots)$

$c = -1$: $f_{-1}(0) = -1$ $f_{-1}(-1) = 0$

$(0, -1, 0, -1, 0, -1, \ldots)$

$c = 0$: $(0, 0, 0, 0, \ldots)$

$c = \frac{1}{4}$: $f_{1/4}\left(\frac{1}{2}\right) = \frac{1}{2}$. $f_{1/2}^{\,n}(0) \longrightarrow \frac{1}{2}$ as $n \to \infty$.

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$c < -2$ or $c > \frac{1}{4}$: $f_c^n(0) \longrightarrow \infty$ as $n \to \infty$.

$c \in [-2, \frac{1}{4}]$: $f_c^n(0) \leq 2$ $\forall n \geq 0$.

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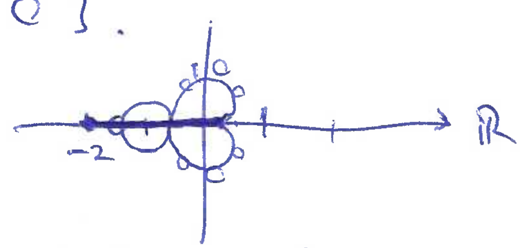
$M = \{ c \in \mathbb{C} \mid |f_c^n(0)| \leq 2 \ \forall n \geq 0 \}$.

$c = i$: $f_i(0) = i$

$\boxed{\text{Note: } i \in \text{ANTENNA!}}$




$f_i(i) = -1 + i$

$f_i(-1+i) = -2i + i = -i$ $(0, i, -1+i, -i, -1+i, -i, \ldots)$

# 640:403 - Introductory Theory of Functions of a Complex Variable

**Section 02, Spring 2018**

**Anders Buch (asbuch ⊗ math • rutgers • edu)**

## Resources:

**Web sites:**
Course web site http://sites.math.rutgers.edu/~asbuch/complex_s18/
Homework and exam scores

**Text:**
Stephen D. Fisher, Complex Variables (2nd edition)

**Syllabus:**
syllabus.pdf

**Lectures:**
Tuesday and Thursday 1:40 - 3:00 PM in SEC-212 (Busch)

**Office hours:**
TBA in HILL-234 (Busch)

## Grading:

Midterm 1, TBA in class, 22%
Midterm 2, TBA in class, 22%
Final Exam, TBA, 44%
Weekly Homework, 12% total.

## Homework Policy:

(1) Late homework is not accepted.

(2) It is fine to discuss the problems with others, but write-ups must be individual. If you have received help for solving a problem, then cite your source(s).

(3) Regard a homework problem as an essay with rigorous mathematical content. Explain what you do without making your explanation longer than necessary. Write neatly. It is your responsibility that whoever reads your work will understand and enjoy it!

(4) STAPLE your work!!!

Assigned homework sets will show up on this course web site.

## Assigned homework:

Homework 0:
Bookmark this page and buy a STAPLER!
Homework 1:
To appear right here!

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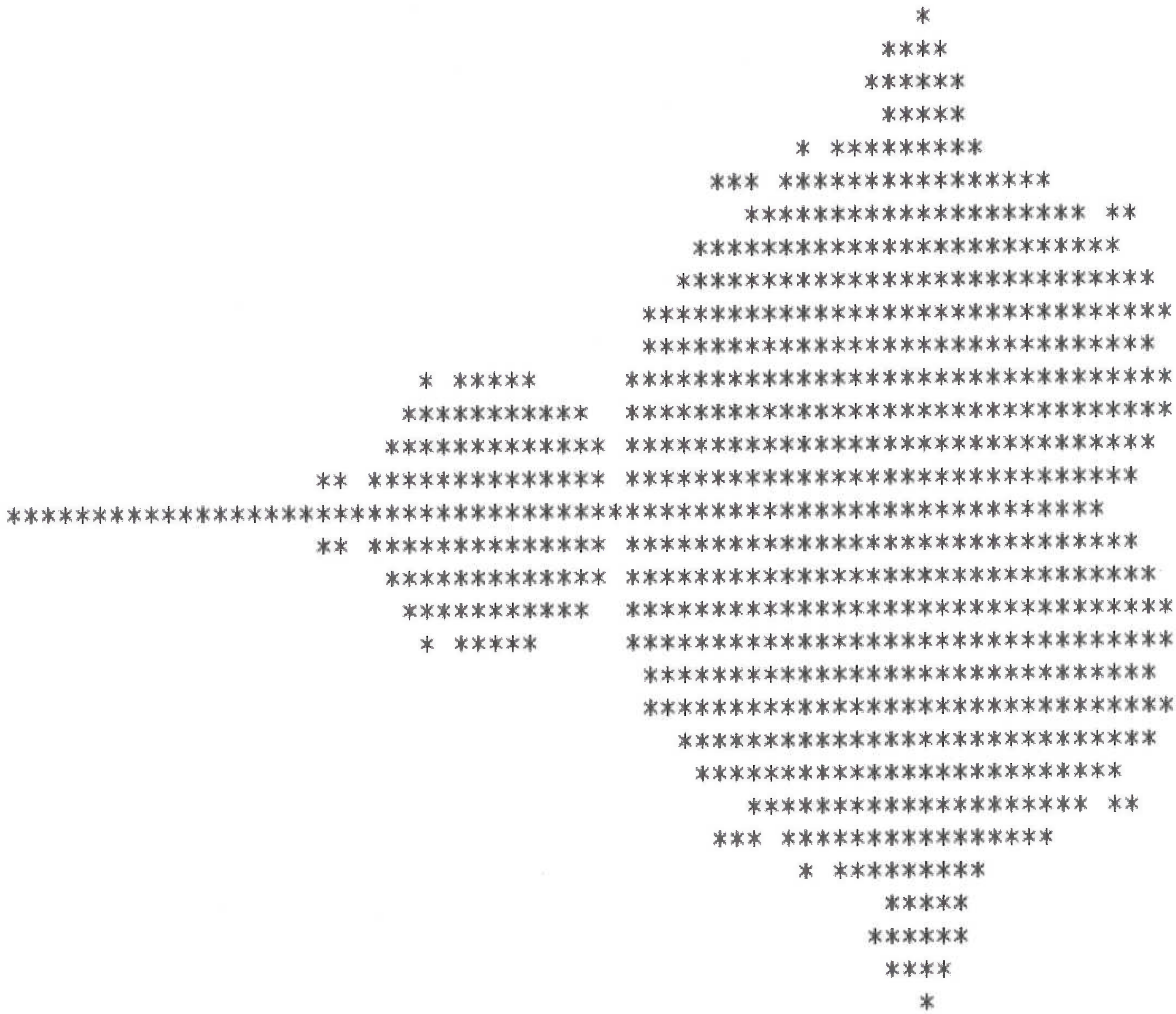
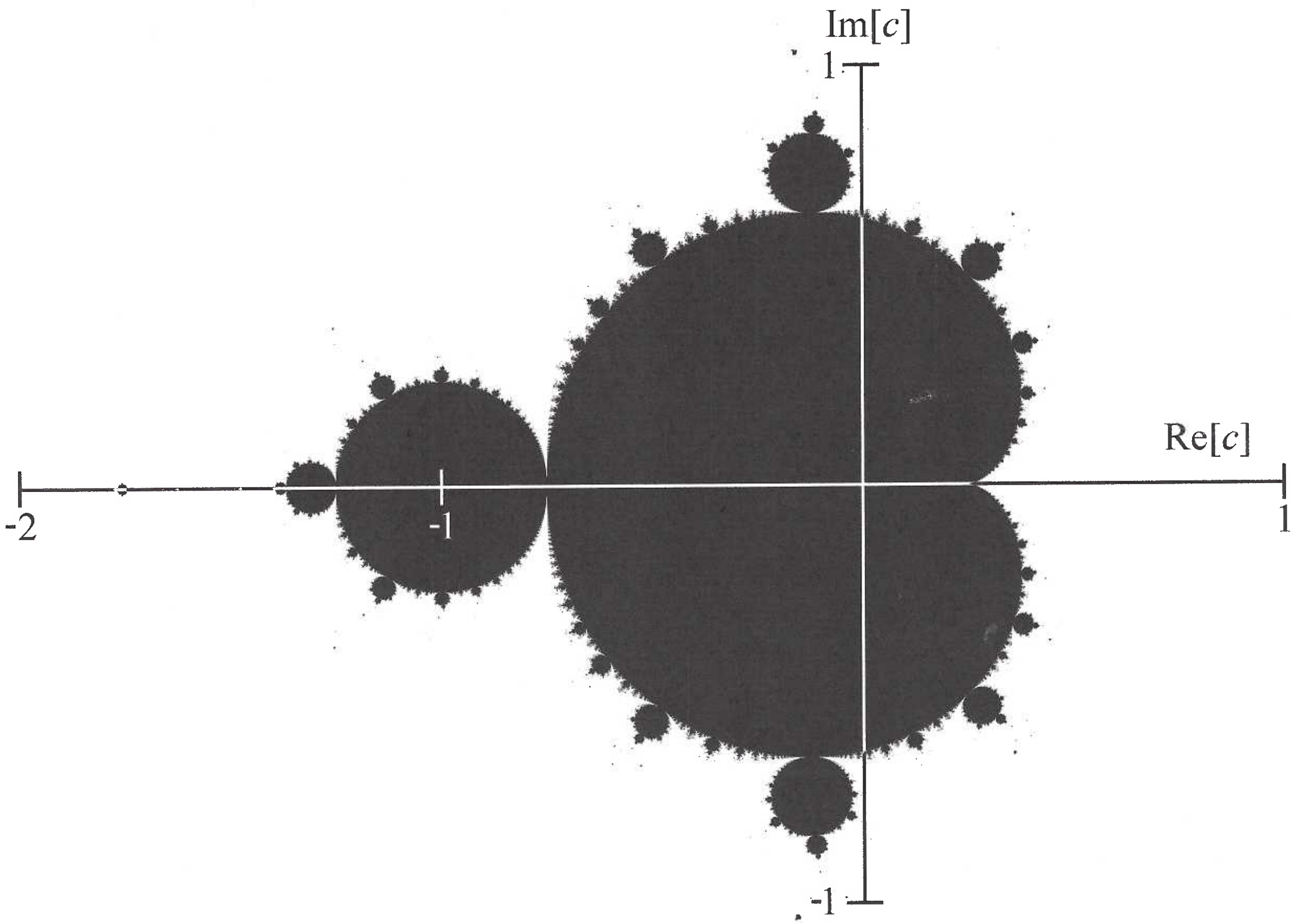
Forgot in 1st class:

- $\mathrm{Arg}: \mathbb{C} - \{0\} \longrightarrow [-\pi, \pi)$ NOT continuous
- $\mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ is a ring homomorphism.
$\overline{zw} = \bar{z}\,\bar{w}$

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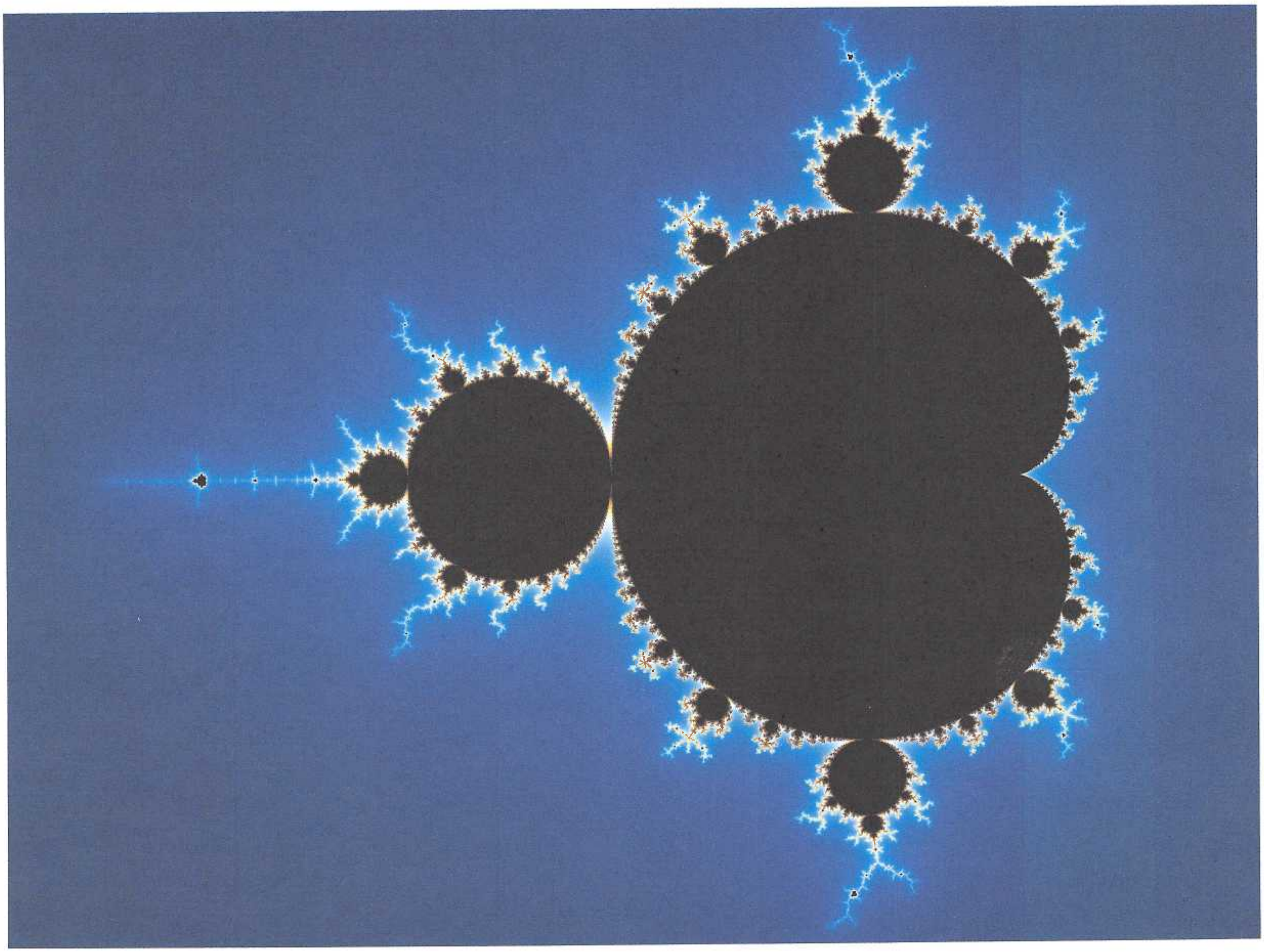

Complex Variables 1/18/2018

## 1.2 Some Geometry

$z = x + iy = (x,y)$. $\bar{z} = x - iy$ complex conjugate.

**Q:** If $z, w \in \mathbb{C}$, is $\bar{z}\,\bar{w} = \overline{zw}$ ? $\theta = \arg(z)$.

Yes! Use $\bar{z} = |z|(\cos(-\theta) + i\sin(-\theta))$,

$\therefore$ The map $f: \mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto \bar{z}$ is a field automorphism

bijective, $f(0) = 0$, $f(z+w) = f(z) + f(w)$, $f(1) = 1$, $f(zw) = f(z)\,f(w)$

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Dot product $z = x + iy$, $w = s + it$

$\vec{z} \bullet \vec{w} = xs + yt = \mathrm{Re}(z\bar{w}) = \mathrm{Re}(\bar{z}w)$

Note: $\mathrm{Re}(z\bar{w}) \leq |z\bar{w}| = |z|\,|w|$. $\boxed{\vec{z} \bullet \vec{w} \leq |\vec{z}| \cdot |\vec{w}|}$


In fact: $\vec{z} \bullet \vec{w} = |\vec{z}| \cdot |\vec{w}| \cos(\alpha)$

$z = |z|(\cos\theta + i\sin\theta)$ $\boxed{\alpha = \theta - \varphi}$

$w = |w|(\cos\varphi + i\sin\varphi)$

$z\bar{w} = |z|\,|w|\,(\cos(\theta - \varphi) + i\sin(\theta - \varphi))$

$\mathrm{Re}(z\bar{w}) = |z|\,|w| \cos(\alpha)$

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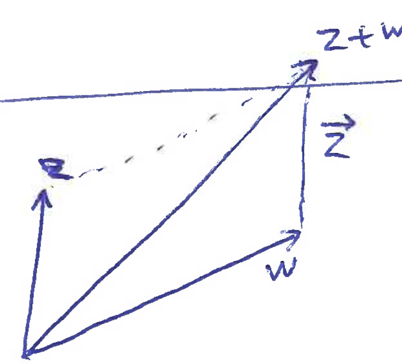
Triangle inequality: $|z + w| \leq |z| + |w|$




$|z+w|^2 = |(x+s) + i(y+t)|^2$

$= (x+s)^2 + (y+t)^2$

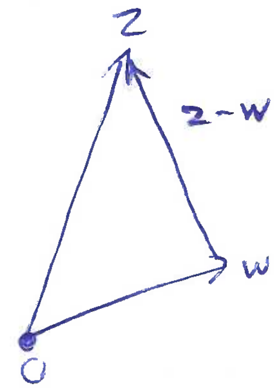
$= x^2 + y^2 + s^2 + t^2 + 2(xs + yt)$

$= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w})$

$\leq |z|^2 + |w|^2 + 2|z|\,|w|$

$= (|z| + |w|)^2$

<u>Alternative version</u> $\big||z| - |w|\big| \leq |z-w|$




Enough: $|z| - |w| \leq |z-w|$

$|z| \leq |z-w| + |w|$

Take $u = z - w$:

$|z| = |u+w| \leq |u| + |w| = |z-w| + |w|.$

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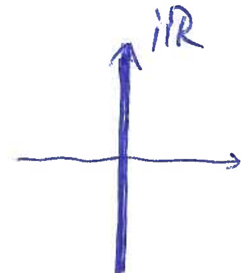
<u>Lines</u>

Equation: $Re(z) = 0.$

Solutions: $\{z \in \mathbb{C} \mid Re(z) = 0\} = i\mathbb{R}$

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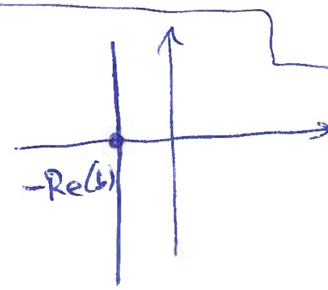
$b \in \mathbb{C}.$ $Re(z+b) = 0$

$Re(z) = -Re(b)$

$\{x = -Re(b)\}.$

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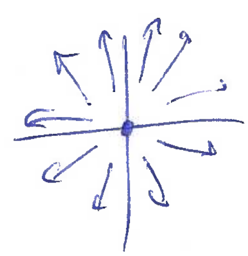
$a, b \in \mathbb{C}$, $a \neq 0$.

$Re(az + b) = 0$ ?

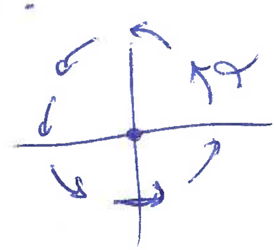
$a = |a|(\cos(\alpha) + i\sin(\alpha))$

Multiply with $|a|$: scale by factor $|a|$



Multiply with $\frac{a}{|a|} = \cos(\alpha) + i\sin(\alpha)$:

Rotate by $\alpha$

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$Re(az+b) = 0$

$\Updownarrow$

$az \in \{x = -Re(b)\}$

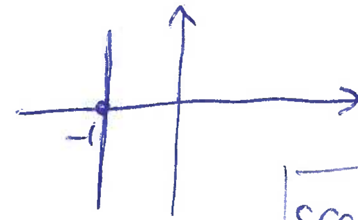
$\Updownarrow$

$\frac{az}{|a|} \in \left\{x = -\frac{Re(b)}{|a|}\right\} \Leftrightarrow z \in \left\{x = -\frac{Re(b)}{|a|}\right\}$ rotated by $-\alpha$.

**Example** $\operatorname{Re}((1+i)z+1) = 0$

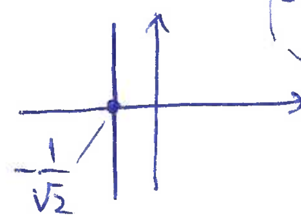
$$a = 1+i = \sqrt{2}\left(\cos\left(\tfrac{3\pi}{4}\right) + i\sin\left(\tfrac{3\pi}{4}\right)\right)$$

$\operatorname{Re}(z+1) = 0$ $\quad\{x = -1\}$



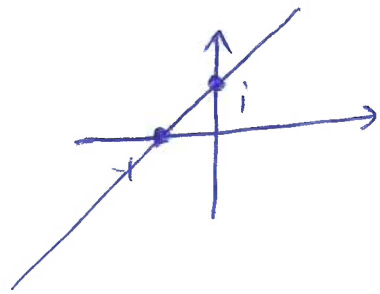

$\operatorname{Re}(\sqrt{2}z+1) = 0$ $\quad\left\{x = -\frac{1}{\sqrt{2}}\right\}$

scale by $1/\sqrt{2}$




$\operatorname{Re}((1+i)z+1) = 0$

Rotate by $-\frac{3\pi}{4}$

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**Roots of complex numbers**

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$$

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

De Moivre: $\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$

Let $w = |w|(\cos\varphi + i\sin\varphi) \in \mathbb{C}$, $\quad w \neq 0$.

Solve $z^n = w$. $\quad$ (i.e. find "$\sqrt[n]{w}$")

$$z = |z|(\cos\theta + i\sin\theta)$$

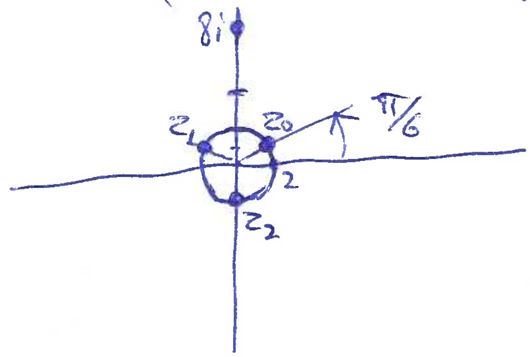
$$z^n = |z|^n(\cos(n\theta) + i\sin(n\theta)) = |w|(\cos\varphi + i\sin\varphi)$$

$$\Rightarrow |z| = \sqrt[n]{|w|} \quad \text{and} \quad n\theta = \varphi + k\cdot 2\pi, \quad k \in \mathbb{Z}$$

$$\theta = \frac{\varphi}{n} + k\,\frac{2\pi}{n}, \quad k \in \mathbb{Z}.$$

$$z = \sqrt[n]{|w|}\,(\cos\theta + i\sin\theta), \quad k \in \mathbb{Z}, \quad 0 \le k < n.$$

Example Find "$\sqrt[3]{8i}$". $z^3 = 8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$




$$z_0 = \sqrt[3]{8}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$$

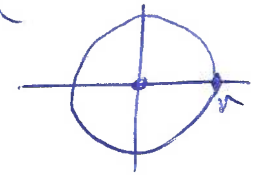
$$= \sqrt{3} + i$$

$$z_1 = \sqrt[3]{8}\left(\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right) = -\sqrt{3} + i$$

$$z_2 = \sqrt[3]{8}\left(\cos\Big(\underbrace{\frac{\pi}{6} + 2\cdot\frac{2\pi}{3}}_{\frac{3\pi}{2}}\Big) + i\sin\Big(\qquad\Big)\right) = -2i$$

## Circles

$|z| = r$ : circle with radius $r$, center $0$




$|z-a| = r$ : circle with radius $r$, center $a$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~~Let $a, b, c \in \mathbb{C}$,~~

**Note:** $x^2 + y^2 + ax + by + c = 0$, $a, b, c \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$.

$$\left(x + \tfrac{1}{2}a\right)^2 + \left(y + \tfrac{1}{2}b\right)^2 = k$$

$$\left(k = -c + \tfrac{1}{4}a^2 + \tfrac{1}{4}b^2\right)$$

$k < 0$ : $\varnothing$

$k = 0$ : $(x,y) = \left(-\tfrac{1}{2}a, -\tfrac{1}{2}b\right)$.

$k > 0$ : circle with radius $\sqrt{k}$, center $\left(-\tfrac{1}{2}a, -\tfrac{1}{2}b\right)$.

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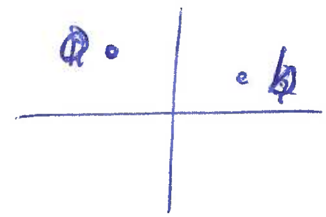
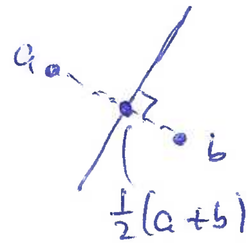
Let $a, b \in \mathbb{C}$. $\rho \in \mathbb{R}$, $\rho > 0$.




Consider $|z-a| = \rho\, |z-b|$

$$\{ z \in \mathbb{C} \mid |z-a| = \rho |z-b| \}$$

Example $\rho = 1$ : Line




Assume $\rho > 0$, $\rho \neq 1$.

$$|z-a|^2 = \rho^2 |z-b|^2 \qquad z = x+iy$$

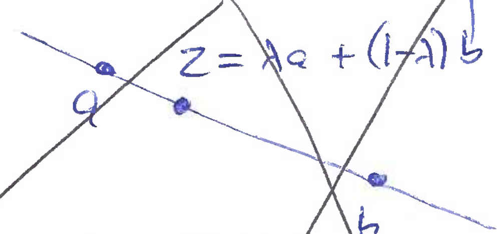
$$(1-\rho^2)\, x^2 + (1-\rho^2)\, y^2 + Ax + By + C = 0$$

$$A, B, C \in \mathbb{R}.$$

Solution set must be circle !!

~~Find 2 points on line through~~ $a, b$.

$$\boxed{z = \lambda a + (1-\lambda) b,} \ \boxed{\lambda \in \mathbb{R}}$$




$$z - a = (\lambda - 1)(a-b)$$

$$z - b = \lambda\, (a-b)$$

$$|z-a|^2 = \rho^2 |z-b|^2$$

$$\Downarrow$$

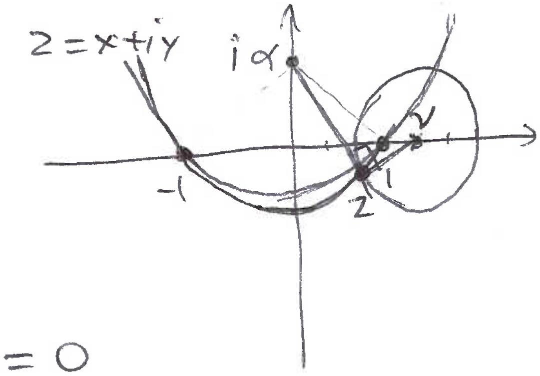
$$(\lambda-1)^2 = \rho\, \lambda^2$$

$$(1-\rho^2)\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1-\rho^2)}}{2(1-\rho^2)}$$

$$\lambda = \frac{1 \pm \rho}{1-\rho^2} = \frac{1}{1 \pm \rho}$$

WLOG $a=1, \quad b=-1.$




$$|z-1|^2 = \rho^2 |z+1|^2$$
$$\Updownarrow$$
$$(x-1)^2 + y^2 = \rho^2\left((x+1)^2+y^2\right)$$
$$\Updownarrow$$
$$x^2-2x+1+y^2 = \rho^2(x^2+2x+1+y^2)$$
$$\Updownarrow$$
$$(1-\rho^2)(x^2+y^2) - 2x(1+\rho^2) + (1-\rho^2) = 0$$
$$\Updownarrow$$
$$x^2+y^2-2\nu x+1 = 0 \quad \star \quad , \qquad \nu = \frac{1+\rho^2}{1-\rho^2}$$
$$\Updownarrow$$
$$(x-\nu)^2+y^2 = \nu^2-1$$

Center: $(\nu, 0)$

Radius: $\sqrt{\nu^2-1} = \left|\frac{2\rho}{1-\rho^2}\right|$

First family of circles: $|z-1|^2 = \rho^2|z+1|^2$

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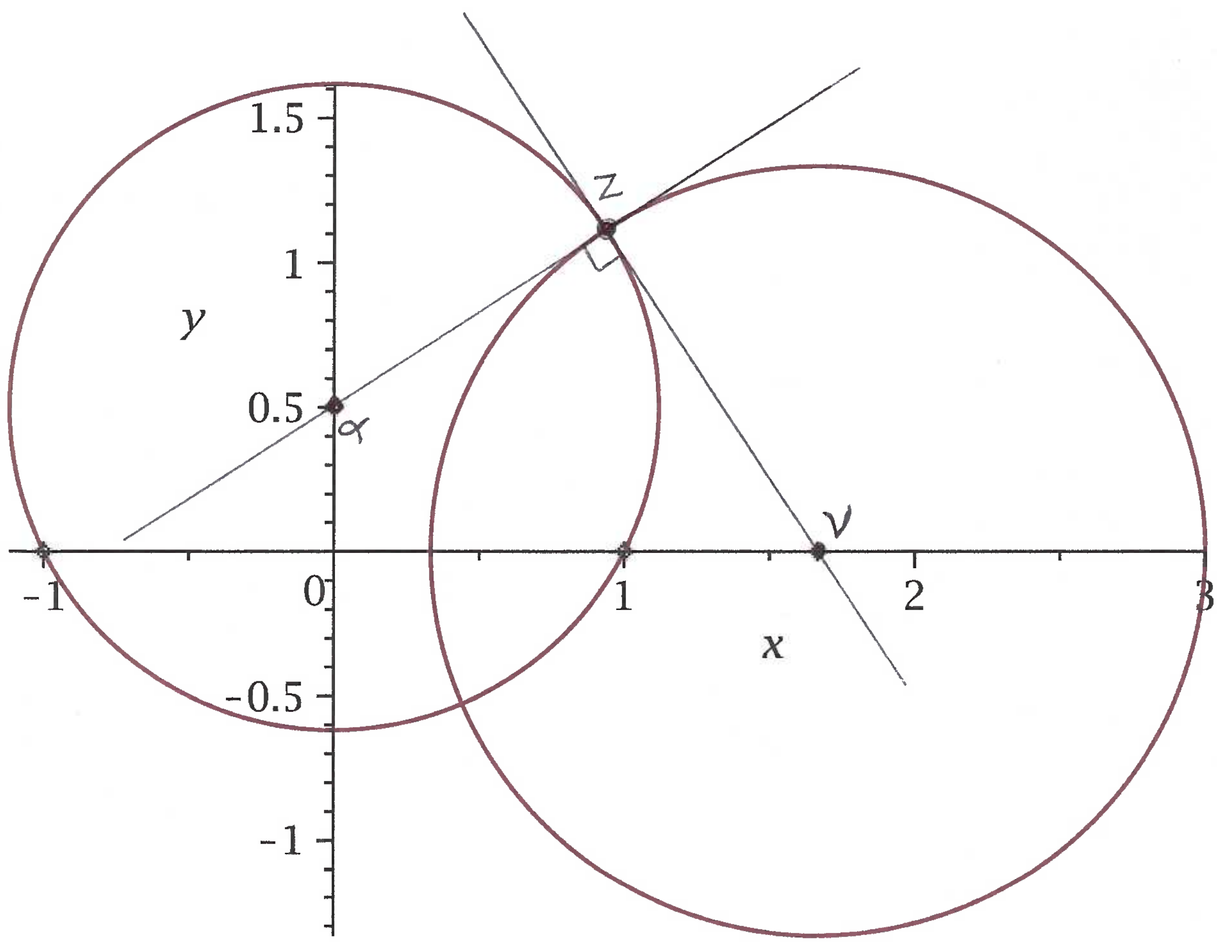
Second family: Center $\overset{i\alpha=}{}(0,\alpha)$, through $a=1, \ b=-1.$

$$|z-i\alpha| = |1-i\alpha|$$
$$\Updownarrow$$
$$x^2+(y-\alpha)^2 = 1+\alpha^2$$
$$\Updownarrow$$
$$x^2+y^2-2y\alpha-1 = 0 \quad \ast$$

Claim: Every circle in first family is perpendicular to every circle in second family.

Equivalent: If $z=x+iy$ is on both circles, then
$$(z-\nu) \perp (z-i\alpha)$$

$$(x-\nu, y)\cdot(x, y-\alpha) = x^2-\nu x+y^2-\alpha y = \tfrac{1}{2}(\star + \ast) = 0$$



1.5
1
0.5
-0.5
-1
y
-1
0
1
2
3
x
z
α
ν