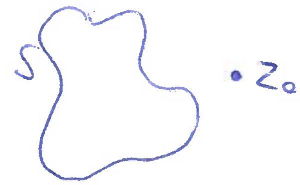
1.3 Subsets of the plane.

$S \subseteq \mathbb{C}$ subset. $z_0 \in \mathbb{C}$.



<u>Def</u> z_0 is a <u>limit point</u> of S
if some sequence of elements of S converge to z_0.
~~[illegible]~~ $(a_n \in S, \quad a_n \to z_0 \text{ as } n \to \infty.)$

<u>Def</u> The closure $\overline{S}$ is the set of all limit points of S.

<u>Examples</u>

- $\mathrm{Re}(z) < 3$

- $S = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0$ and $\mathrm{Re}(z) > 0\}$

- $S = \{z \in \mathbb{C} \mid |z| < 1\}$

- $S = \{z \in \mathbb{C} \mid |z| = 1 \text{ and } \mathrm{Im}(z) > 0\}$

- $S = \mathbb{Q} \subseteq \mathbb{C}$

- $S = \mathbb{Q} + i\mathbb{Q} = \{x + iy \mid x, y \in \mathbb{Q}\}$

- $S = \mathbb{Z} + i\mathbb{Z} = \{x + iy \mid x, y \in \mathbb{Z}\}$

<u>Def</u> S is <u>closed</u> in $\mathbb{C}$ if $S = \overline{S}$

~~S is open in $\mathbb{C}$ if $\mathbb{C} - S$ is closed.~~

~~Note S is closed if all limit points of S are contained in S.~~

~~S is open if no limit points of $\mathbb{C} - S$ are in S.~~

<u>Example</u> If $S \subseteq \mathbb{C}$ any subset, then $\overline{S}$ is closed.
$$\overline{\overline{S}} = \overline{S}.$$

Def $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$.

z_0 is an interior point of S if z_0 NOT limit point of $\mathbb{C} \setminus S$.

Examples All of the above.

$\boxed{\text{Note: } z_0 \in S}$

Next: interior point of S = point at safe distance from $\mathbb{C} \setminus S$.

---

open ball $B(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}$

Thm z_0 is an interior point of S

$\iff \exists\, r > 0 : B(z_0, r) \subseteq S$

Proof

$\Leftarrow$: If $B(z_0, r) \subseteq S$ then no sequence in $\mathbb{C} \setminus S$ can converge to z_0.

$\Rightarrow$: Assume $(\exists\, r > 0 : B(z_0, r) \subseteq S)$ is false.

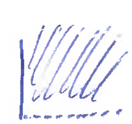
Equiv: $\forall\, r > 0 : B(z_0, r) \not\subseteq S$.

For each $n \in \mathbb{N}$, choose $a_n \in B(z_0, \frac{1}{n}) \setminus S$.

Then $a_n \longrightarrow z_0$, so z_0 limit point of $\mathbb{C} \setminus S$

i.e. z_0 NOT interior point of S.

Def A subset $S \subseteq \mathbb{C}$ is **open** if all points of S are interior points of S.



Example $S = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0,\ \mathrm{Im}(z) > 0 \}$

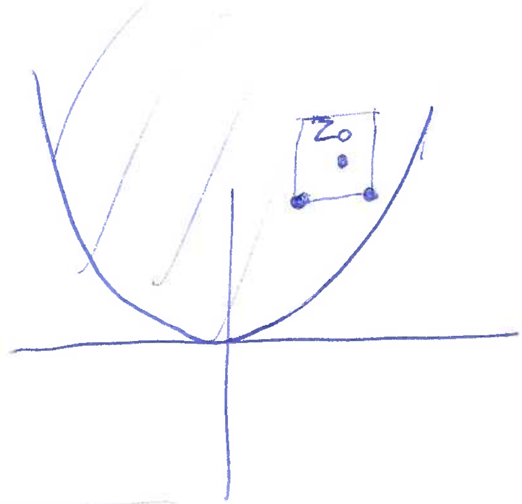
$z_0 = \frac{1}{3} + \frac{1}{4} i$ | $z_0 = \frac{1}{10} + 7i$ | $z_0 = 7i$

Examples of open sets.

1) $B(z_0, R)$

2) $S = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$

3) $S = \{x + iy \mid x^2 < y\}$



$z_0 = x_0 + iy_0 \in S$.

$y_0 > x_0^2$

Choose $r > 0$ so small that

$r^2 - 2x_0 r + r < y_0 - x_0^2$

$r^2 + 2x_0 r + r < y_0 - x_0^2$

Choose $r > 0$ such that

$(x_0 - r, y_0 - r) \in S$ and

$(x_0 + r, y_0 - r) \in S$.

$(x_0 - r)^2 < y_0 - r$

$(x_0 + r)^2 < y_0 - r$

$r^2 - 2x_0 r + r < y_0 - x_0^2$

$r^2 + 2x_0 r + r < y_0 - x_0^2$

~~Then~~

Claim: $B(z_0, r) \subseteq S$.

Let $z = x + iy \in B(z_0, r)$

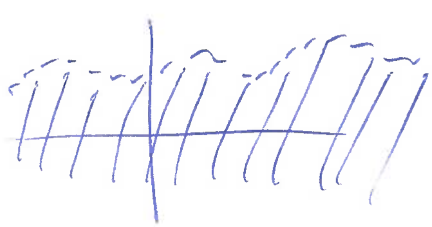
Then $|x| \leq |x_0| + r$

$x^2 \leq (|x_0| + r)^2 < y_0 - r < y$

So $z = x + iy \in S$.

4) $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous function.

$S = \{x + iy \mid f(x) > y\}$ is open.



Example

Let U_1 and U_2 be open subsets of $\mathbb{C}$.

Then $U_1 \cap U_2$ and $U_1 \cup U_2$ are open.

Boundary point

$S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$.

z_0 is boundary point of S

$\Updownarrow$

$z_0 \in \overline{S} \cap \overline{(\mathbb{C}-S)}$ — limit point of both S, $\mathbb{C}-S$

$\Updownarrow$

$\forall r>0: \quad B(z_0,r)\cap S \neq \emptyset$ and $B(z_0,r)\cap(\mathbb{C}-S)\neq\emptyset$

— no safe distance to S or S^c.

~~∂S = set of~~

$\partial S = \overline{S} \cap \overline{\mathbb{C}-S}$ set of boundary points of S.

Examples: • $B(0,1)$ • $\{\mathrm{Re}(z)>0\}$ • $\mathbb{C}$ • $\mathbb{Q}+i\mathbb{Q}$ • $\mathbb{Z}+i\mathbb{Z}$

Thm

S is closed $\iff \partial S \subseteq S$

S is open $\iff \partial S \cap S = \emptyset$

S is open $\iff S^c$ is closed

Example $\mathbb{C}$ both open & closed in $\mathbb{C}$.

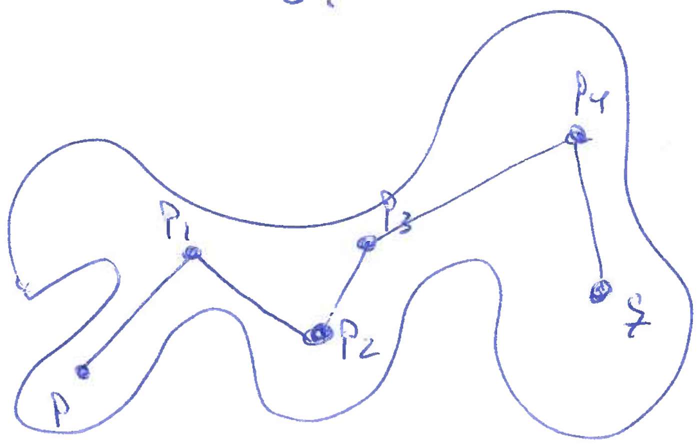
Open connected set

An open set $S \subseteq \mathbb{C}$ is <u>connected</u> if $\forall\, p, q \in S$ $\exists$ <u>polygonal curve</u> from p to q inside S.

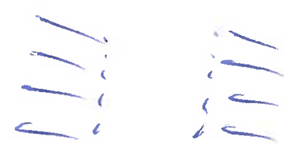
That is: $\exists\, P_1, P_2, \dots, P_m \in S$ such that all line segments

pP_1, P_1P_2, $\dots$, $P_{m-1}P_m$, P_mq

are contained in S.

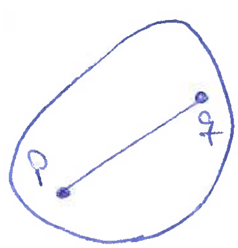


<u>Non-example:</u> $S = \{ z \in \mathbb{C} \mid |\mathrm{Re}(z)| > 1 \}$

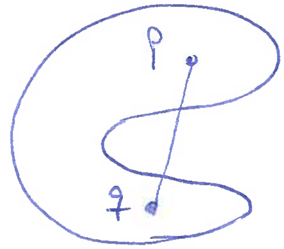


<u>Def</u> $S \subseteq \mathbb{C}$ is <u>convex</u> if

$\forall\, p, q \in S$, the line segment $pq \subseteq S$.



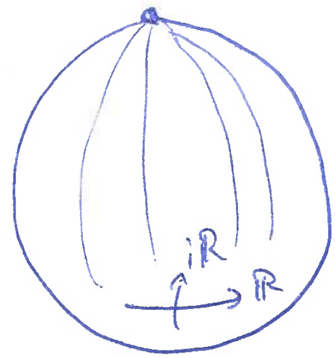
<u>Non-example</u>



Riemann Sphere

$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ — add point at ∞.

Given $\{a_n\}$, $a_n \in \mathbb{C}$, we ~~say~~ "agree" that
$a_n \to \infty$ if $|a_n| \to \infty$ as $n \to \infty$.



$S \subseteq \mathbb{CP}^1$.

∞ is a limit point of S
if $\infty \in S$ or $|a_n| \to \infty$
for some seq. (a_n) in $S \subseteq \mathbb{C}$.

∞ interior point of S if not limit point of $\mathbb{CP}^1 - S$.

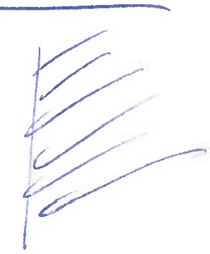
open ball of cofe distance :
$$B(\infty, r) = \{z \in \mathbb{C} \mid |z| > \tfrac{1}{r}\} \cup \{\infty\}.$$

Bijection : $f : \mathbb{CP}^\infty - \{0\} \longrightarrow \mathbb{C}$
$$f(z) = \frac{1}{z}$$
$$f(\infty) = 0.$$

Note : $a_n \to \infty$ in $\mathbb{CP}^1$ $\Leftrightarrow$ $f(a_n) \to 0$ in $\mathbb{C}$
$\frac{1}{a_n} \to 0$.

Example $S = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\} \subseteq \mathbb{CP}^1$.

Then $\infty \in \partial S = \overline{S} \cap \overline{\mathbb{CP}^1 - S}$.



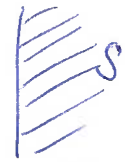
Complex Variables 1/25/2018 $\boxed{\text{OFFICE HOURS}}$

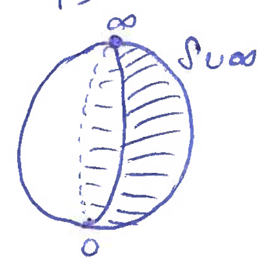
Rieman Sphere $\mathbb{P}^1 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

Given (a_n), $a_n \in \mathbb{C}$, we "agree" that
$a_n \longrightarrow \infty$ iff $|a_n| \to +\infty$ as $n \to \infty$.

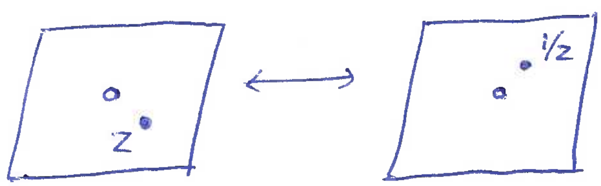
Def ∞ is a limit point of $S \subseteq \mathbb{CP}^1$ if
$\infty \in S$ or $|a_n| \longrightarrow +\infty$ for some seq. (a_n), $a_n \in S \cap \mathbb{C}$.

Example $S = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \geq 0\}$ is closed in $\mathbb{C}$.
Closure of S in $\mathbb{P}^1$: $S \cup \{\infty\}$.





Construction: Glue two copies of $\mathbb{C}$ by identifying $z \in \mathbb{C} \setminus \{0\}$ in one copy with $\frac{1}{z} \in \mathbb{C} \setminus \{0\}$ in other copy.



$$z \longmapsto w = 1/z$$

$$z = 1/w \longleftarrow w$$

Formally: Equiv. relation $\sim$ on $\mathbb{C} \times \{0,1\}$

$$(z,s) \sim (w,t) \iff zw = 1 \text{ and } s \neq t$$

$$\mathbb{CP}^1 = \left(\mathbb{C} \times \{0,1\}\right)/\sim$$

$$\mathbb{C} = \{(z,0) \in \mathbb{CP}^1 \mid z \in \mathbb{C}\}.$$

$$\infty = (0,1)$$

Note: $S \subseteq \mathbb{C}$. S open in $\mathbb{C}$
$\iff$ S open in $\mathbb{P}^1$
($\iff$ all points of S are interior points.)

Note: $S \subseteq \mathbb{P}^1$. ∞ is interior point of S $\iff$ Not limit point of $\mathbb{P}^1 \setminus S$
$\iff$ S contains open ball of some distance r:
$B(\infty, r) \subseteq S$ where $B(\infty, r) = \{z \in \mathbb{C} \mid |z| > 1/r\} \cup \{\infty\}$.

1.4 Functions and Limits

Let $f: D \to \mathbb{C}$ be a function, where $D \subseteq \mathbb{C}$ subset.

Means: For every $z = x + iy \in D$, f gives us $w = f(z) \in \mathbb{C}$.

Domain of f: D

Range of f: $\{ f(z) \mid z \in D \} \subseteq \mathbb{C}$.

Example

$f(z) = \dfrac{1}{\mathrm{Im}(z)}$. Can use ~~be defined on~~ any domain $D \subseteq \mathbb{C} \setminus \mathbb{R}$.

Range: $\{ f(z) \mid z \in D \} \subseteq \mathbb{R} \setminus \{0\}$.

Example $f(z) = z^3$, $z \in \mathbb{C}$.

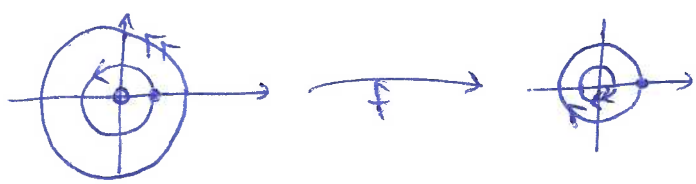
$z = r(\cos\theta + i\sin\theta)$

$f(z) = r^3(\cos 3\theta + i \sin 3\theta)$

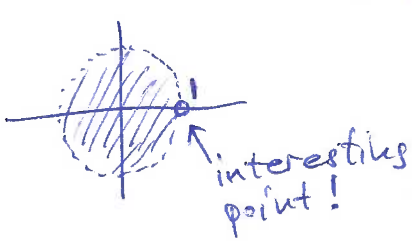
Visualize:

Example $f(z) = \frac{1}{z}$, Domain $\mathbb{C} \setminus \{0\}$

$f(z) = \frac{1}{r}\left(\cos(-\theta) + i \sin(-\theta)\right)$



Example $f(z) = \dfrac{1+z}{1-z}$, $z \in B(0,1)$

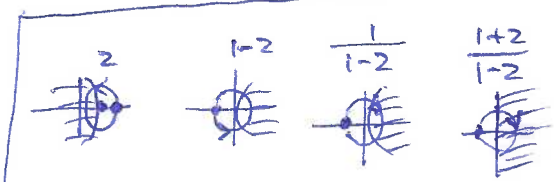


$$f(z) = \frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})} = \frac{1 - |z|^2 + 2i\,\mathrm{Im}(z)}{|1-z|^2}$$

$z \in B(0,1) \Rightarrow \mathrm{Re}(f(z)) > 0$.

Range: $\{ f(z) \mid z \in B(0,1) \} = \{ w \in \mathbb{C} \mid \mathrm{Re}(w) > 0 \}$

CHECK!

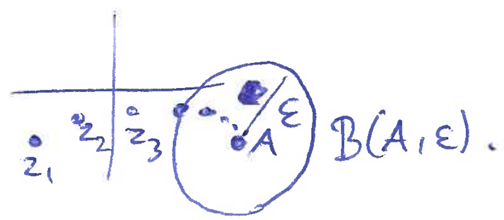


Limits $\{z_n\}_{n=1}^{\infty}$ seq. of complex numbers. $A \in \mathbb{C}$.

Def $\{z_n\}$ converges to A, written $\lim_{n\to\infty} z_n = A$, if:

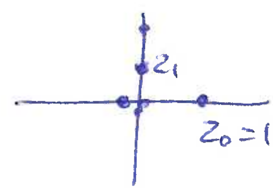
$$\forall \varepsilon > 0 \ \exists N > 0 : \quad |z_n - A| < \varepsilon \quad \text{for all } n \geq N.$$

Equivalent: For any open ball $B(A, \varepsilon)$ with center A, only finitely many z_n are outside $B(A, \varepsilon)$!!



Example $z_n = \left(\frac{i}{2}\right)^n$

$z_n \to 0$ as $n \to \infty$.



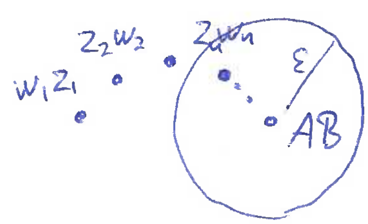
Note $z_n \to A$ in $\mathbb{C}$ as $n \to \infty$ $\iff$ $|z_n - A| \to 0$ in $\mathbb{R}$ as $n \to \infty$.

Thm Let $\{z_n\}$ and $\{w_n\}$ be sequences in $\mathbb{C}$. Assume $z_n \to A$ and $w_n \to B$ as $n \to \infty$. Then

(1) $z_n + w_n \longrightarrow A + B$

(2) $z_n w_n \longrightarrow AB$

(3) If $B \neq 0$ then $\frac{1}{w_n} \longrightarrow \frac{1}{B}$.



Proof of (2): Let $\varepsilon > 0$ be given.

Want $|z_n w_n - AB| < \varepsilon$.

$$
\begin{aligned}
|z_n w_n - AB| &= |z_n w_n - z_n B + z_n B - AB| \\
&\leq |z_n w_n - z_n B| + |z_n B - AB| \\
&= |z_n| \cdot |w_n - B| + |z_n - A| \cdot |B|
\end{aligned}
$$

If $|z_n - A| < r$ and $|w_n - B| < r$ then

$$|z_n w_n - AB| < (|A|+r)r + r|B| = r(r+|A|+|B|)$$

Choose $r > 0$ such that $r(r+|A|+|B|) < \varepsilon$.

Choose $N > 0$ such that $|z_n - A| < r$ and $|w_n - B| < r$ for all $n > N$.

Then $|z_n w_n - AB| < r(r+|A|+|B|) < \varepsilon \quad \forall n > N$.

$\square$

<u>Example</u> $$z_n = \frac{\frac{1}{n}\sin(n) + \frac{i-n}{i+n}}{i + 2^{-n}}$$

$\frac{1}{n}\sin(n) \to 0$

$$\frac{i-n}{i+n} = \frac{i/n - 1}{1/n + 1} \longrightarrow \frac{0-1}{0+1} = -1$$

$i + 2^{-n} \longrightarrow i$

$z_n \longrightarrow \frac{-1}{i} = i$ as $n \to \infty$.

<u>Limit of function</u>

$f : S \longrightarrow \mathbb{C}$, $S \subseteq \mathbb{C}$.

Let $z_0 \in \overline{S} = S \cup \partial S$

<u>Def</u> f has <u>limit</u> $L \in \mathbb{C}$ at z_0 if

$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \;\; |f(z) - L| < \varepsilon$ for all $z \in S$ with $|z - z_0| < \delta$.

<u>Equivalent:</u> Given any open ball $B(L, \varepsilon)$ around L, can find open ball $B(z_0, \delta)$ around z_0, such that f maps $B(z_0, \delta) \cap S$ into $B(L, \varepsilon)$.

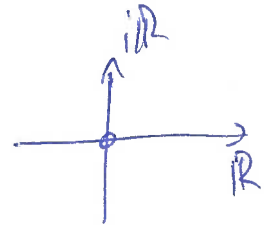
<u>Example</u> $f(z) = \dfrac{z^3-1}{z-1}, \ z \neq 1$ $\qquad f(z) = \dfrac{(z^2+z+1)(z-1)}{z-1}$

$$\lim_{z \to 1} f(z) = 1+1+1 = 3.$$

<u>Example</u> $f(z) = \dfrac{\bar{z}}{z}$ has no limit at $z=0$.

$f(x) = \dfrac{x}{x} = 1$ for all $x \in \mathbb{R}$

$f(iy) = \dfrac{-iy}{iy} = -1$ for all $y \in \mathbb{R}$.



Limit at ∞

Assume ∞ limit point of $S \subseteq \mathbb{CP}^1$, $f : S \longrightarrow \mathbb{C}$.

<u>Def</u> f has limit $L \in \mathbb{C}$ at $z_0 = \infty$ if

$\forall \varepsilon > 0 \ \exists M > 0 : |f(z) - L| < \varepsilon$ for all $z \in S$ with $|z| \geq M$.

Equivalent:

$f(1/w) \longrightarrow L$ as $w \longrightarrow 0$

<u>Example</u>

$f(z) = \dfrac{3z^2-1}{z^2+z}$ $\qquad f(z) \longrightarrow 3$ as $z \to \infty$.

$$f(1/w) = \frac{3w^{-2}-1}{w^{-2}+w^{-1}} = \frac{3-w^2}{1+w} \longrightarrow 3 \text{ as } w \to 0$$

<u>Thm</u> $f, g : S \longrightarrow \mathbb{C}$, $S \subseteq \mathbb{CP}^1$, z_0 lim. pt. of S.

Assume $f(z) \longrightarrow A$, $g(z) \longrightarrow B$ as $z \to z_0$.

Then $f(z) + g(z) \longrightarrow A+B$

$f(z)\, g(z) \longrightarrow AB$

If $B \neq 0$ then $1/g(z) \to \dfrac{1}{B}$ as $z \to z_0$.

Continuous function

$f : S \longrightarrow \mathbb{C}, \qquad S \subseteq \mathbb{CP}^1.$

f is continuous at $z_0 \in S$ if

$$f(z) \to f(z_0) \quad \text{as} \quad z \to z_0$$

f is continuous on S if cont. at all $z_0 \in S$.

Example
$$f(z) = \begin{cases} e^{-|z|} & \text{if } z \in \mathbb{C} \\ 0 & \text{if } z = \infty. \end{cases}$$

$f : \mathbb{CP}^1 \longrightarrow \mathbb{R}$ continuous.

Infinite series

$$\sum_{k=1}^{\infty} z_n$$

n-th partial sum $\quad S_n = \sum_{j=1}^{n} z_j$

converge $\quad S_n \to S$ as $n \to \infty$

diverge $\quad \{S_n\}$ diverges.

Geometric series

$$\sum_{k=0}^{\infty} \alpha^k = 1 + \alpha + \alpha^2 + \cdots = \begin{cases} \frac{1}{1-\alpha} & \text{if } |\alpha| < 1 \\ \text{diverges} & \text{if } |\alpha| \geq 1. \end{cases}$$

$$S_n = \sum_{k=0}^{n} \alpha = 1 + \alpha + \cdots + \alpha^n$$

$$(1-\alpha) S_n = 1 - \alpha^{n+1}$$

$$S_n = \frac{1-\alpha^{n+1}}{1-\alpha} \longrightarrow \frac{1}{1-\alpha} \quad \text{if } |\alpha| < 1.$$

1.5 Exp, Log, Trig. fcns.

$\boxed{\text{Comments so far?}}$

$z = x + iy \in \mathbb{C}$.

Def $\exp(z) = e^z = e^x(\cos(y) + i\sin(y))$

Note: If $w = s + it$ then

$$e^z e^w = e^x(\cos y + i \sin y) \cdot e^s(\cos t + i \sin t)$$
$$= e^{x+s}(\cos(y+t) + i\sin(y+t))$$
$$= e^{(x+s) + i(y+t)} = e^{z+w}.$$

Other familiar properties:

$$\exp'(z) = \exp(z) \quad \text{and} \quad \exp(0) = 1$$
$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Examples

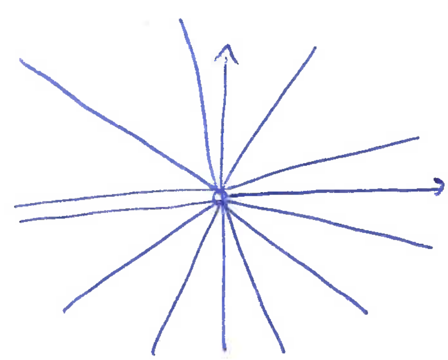
$$|e^z| = e^x = e^{\mathrm{Re}(z)}$$
$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

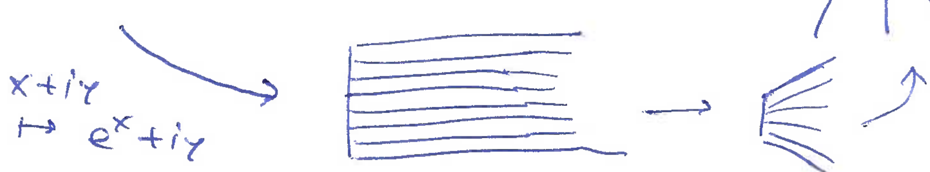
$$\boxed{e^{i\pi} + 1 = 0}$$

Polar coordinates: $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Geometry $D = \{x + iy \mid -\pi < y < \pi\}$

$\exp : D \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\le 0}$





~~Solve $e^z = 1$:~~

~~$e^x = |e^z| = 1 \Rightarrow x = 0$~~

~~$e^{iy} = \cos(y) + i\sin(y) = 1 \Rightarrow y = 2n\pi,\ n \in \mathbb{Z}.$~~

Solve $e^z = w$:

$$e^x = |e^z| = |w| \Rightarrow x = \ln|w|$$

$$e^{iy} = \frac{e^z}{e^x} = \frac{w}{|w|} \Rightarrow y = \arg(w) = \mathrm{Arg}(w) + 2n\pi$$

$n \in \mathbb{Z}.$

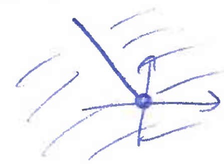
Def $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$ **Def** on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$\log(z)$ is any $a \in \mathbb{C}$ with $e^a = z$.

So $\log(z) = \mathrm{Log}(z) + 2\pi n i,\ n \in \mathbb{Z}$

$$\log(z) = \ln|z| + i\arg(z)$$

Note: We can define continuous incarnation of $\log(z)$ on any region of the form



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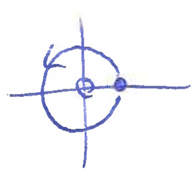
but NOT on all of $\mathbb{C} \setminus \{0\}$.



$\gamma(t) = e^{it},\quad 0 \leq t \leq 2\pi$

$\log(\gamma(t))$ cont.

$\log(\gamma(t)) = it$

$\log \gamma(0) \neq \log \gamma(2\pi).$

but $\gamma(0) = 1 = \gamma(2\pi).$

<u>Def</u> For $a, z \in \mathbb{C}$, define

$$a^z = \exp(z \log(a))$$

<u>Note</u>: NOT WELL DEFINED !!!

But a^n is well def for $n \in \mathbb{Z}$.

$$a^n = \exp(n(\mathrm{Log}(a) + 2\pi i m))$$

<u>Example</u> $i^i = ?$

$$\log(i) = i\frac{\pi}{2} + 2n\pi i \quad , \quad n \in \mathbb{Z}.$$

$$i^i = \exp(i \log(i)) = \exp(i^2 \tfrac{\pi}{2} + 2n\pi i^2)$$

$$= \exp(-\tfrac{\pi}{2} - 2n\pi) \quad , \quad n \in \mathbb{Z}.$$

<u>Note</u>: $i^i \in \mathbb{R}_+$!!

<u>Cool formula</u>

$$e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n \quad , \quad z \in \mathbb{C}.$$

<u>Proof</u>

$$n \,\mathrm{Log}\left(1 + \frac{z}{n}\right) = n \cdot \ln\left|1 + \frac{z}{n}\right| + in\,\mathrm{Arg}\left(1 + \frac{z}{n}\right)$$

$$z = x + iy$$

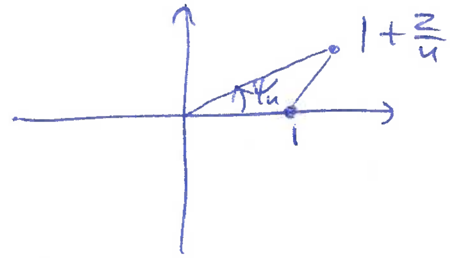
$$n \cdot \ln\left|1 + \frac{z}{n}\right| = \frac{n}{2} \ln\left(\left(1 + \frac{z}{n}\right)\left(1 + \frac{\bar{z}}{n}\right)\right)$$

$$= \frac{n}{2} \ln\left(1 + \frac{x^2+y^2}{n^2} + \frac{2x}{n}\right)$$

l'Hôpital $\Rightarrow$ $n \cdot \ln\left|1 + \frac{z}{n}\right| \longrightarrow x$ as $n \to \infty$

Set $\psi_n = \mathrm{Arg}\left(1+\frac{z}{n}\right)$.

$$\tan(\psi_n) = \frac{y/n}{1+\frac{x}{n}}$$



$$n\,\mathrm{Arg}\left(1+\frac{z}{n}\right) = n\,\psi_n = n\tan(\psi_n)\,\frac{\psi_n}{\tan(\psi_n)}$$

$$= \frac{y}{1+\frac{x}{n}}\cdot\frac{\psi_n}{\tan(\psi_n)}$$

$$n\,\mathrm{Arg}\left(1+\frac{z}{n}\right) \longrightarrow y \quad \text{as } n\to\infty.$$

$$\therefore \quad n\,\mathrm{Log}\left(1+\frac{z}{n}\right) \longrightarrow x+iy = z \quad \text{as } n\to\infty.$$

$$\left(1+\frac{z}{n}\right)^n = \exp\left(n\,\mathrm{Log}\left(1+\frac{z}{n}\right)\right) \longrightarrow \exp(z) \quad \text{as } n\to\infty.$$

□

<u>Trigonometric functions</u>

$$\cos(z) = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$$

<u>Check</u>: usual values for $z\in\mathbb{R}$.

$$\cos(z+2n\pi) = \cos(z) \qquad\qquad \cos(z) = \sin\left(\tfrac{\pi}{2}-z\right)$$

$$\sin(z+2n\pi) = \sin(z) \qquad\qquad \cos(z)^2+\sin(z)^2 = 1.$$

$$\sin(-z) = -\sin(z)$$

$$\sin(\bar{z}) = \overline{\sin(z)} \quad \text{– since } \exp(\bar{z}) = \overline{\exp(z)}$$