Complex Variables 2/1/2018

1.5 Trig. fcns

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \quad z \in \mathbb{C}.$$

$$x \in \mathbb{R}: \quad \cos(x) = \frac{1}{2}\left(\cos(x) + i\sin(x) \;+\; \cos(-x) + i\sin(-x)\right)$$

- same as usual.

$$z \in \mathbb{C}: \quad \cos\left(\frac{\pi}{2} - z\right) = \frac{1}{2}\left(e^{-iz + \frac{\pi}{2}i} + e^{iz - \frac{\pi}{2}i}\right)$$

$$= \frac{1}{2}\left(i e^{-iz} - i e^{iz}\right)$$

$$= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \sin(z).$$

$$\cos(z)^2 + \sin(z)^2 = \frac{1}{4}\left(e^{iz} + e^{-iz}\right)^2 - \frac{1}{4}\left(e^{iz} - e^{-iz}\right)^2 = 1$$

$$\cos(-z) = \cos(z)$$

$$\sin(-z) = -\sin(z)$$

$$\sin(\bar{z}) = \overline{\sin(z)} \qquad \text{since} \quad \exp(\bar{z}) = \overline{\exp(z)}$$

$$\cos(\bar{z}) = \overline{\cos(z)}$$

$$\cos(z + 2n\pi) = \cos(z), \qquad \sin(z + 2n\pi) = \sin(z), \quad n \in \mathbb{Z}$$

Real / Im parts

$$\cos(x+iy) = \frac{1}{2}\left(e^{ix}e^{-y} + e^{-ix}e^{y}\right)$$

$$= \cos(x)\,\frac{1}{2}\left(e^{y} + e^{-y}\right) - i\sin(x)\,\frac{1}{2}\left(e^{y} - e^{-y}\right)$$

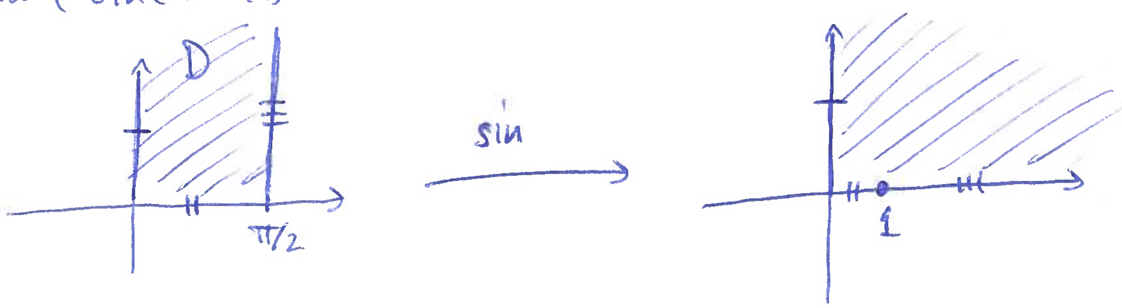
$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

Restrict $\sin(z)$ to $D = \{x+iy \mid 0 < x < \frac{\pi}{2},\ y > 0\}$

$\mathrm{Re}(\sin(x+iy)) = \sin(x)\cosh(y) > 0$

$\mathrm{Im}(\sin(x+iy)) = \cos(x)\sinh(y) > 0$



$\sin(iy) = i\sinh(y)$

$\sin(x) \in [0,1]$ for $x \in [0, \pi/2]$

$\sin(\pi/2 + iy) = \cosh(y) \in [1, \infty)$ for $y \geq 0$.

Check: $\sin(z)$ is injective on D.

Onto first quadrant: Let $w = s + it$, $s, t > 0$.

Choose $c \in (0,1)$ s.t. $\dfrac{s^2}{1-c^2} - \dfrac{t^2}{c^2} = 1$

Choose $x \in (0, \pi/2)$ s.t. $\cos(x) = c$

Then $\dfrac{s^2}{\sin(x)^2} - \dfrac{t^2}{\cos(x)^2} = 1$

Choose $y > 0$ s.t. $\sinh(y) = \dfrac{t}{\cos(x)}$

$\cosh(y)^2 = 1 + \sinh(y)^2$

$= 1 + \dfrac{t^2}{c^2} = \dfrac{s^2}{1-c^2} = \dfrac{s^2}{\sin(x)^2}$

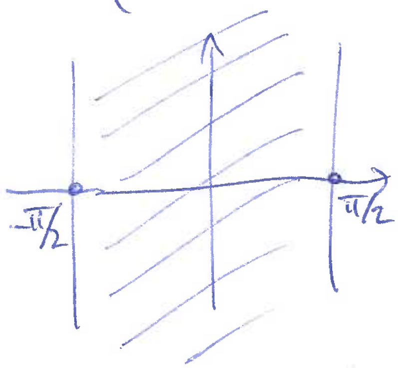
$\cosh(y) = \dfrac{s}{\sin(x)}$.

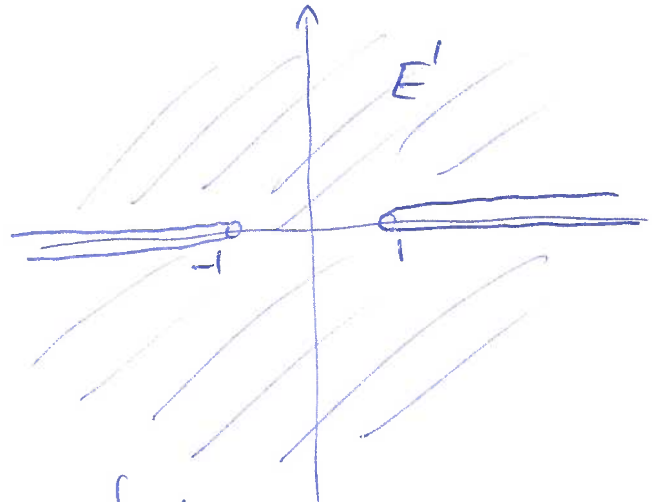
$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) = s + it = w$

Use $\sin(-z) = -\sin(z)$

$\sin(\bar{z}) = \overline{\sin(z)}$:

$$D' = \left\{ x+iy \;\middle|\; -\frac{\pi}{2} < x < \frac{\pi}{2} \right\} \qquad E' = \left\{ s+it \;\middle|\; t \neq 0 \text{ OR } |s| < 1 \right\}$$





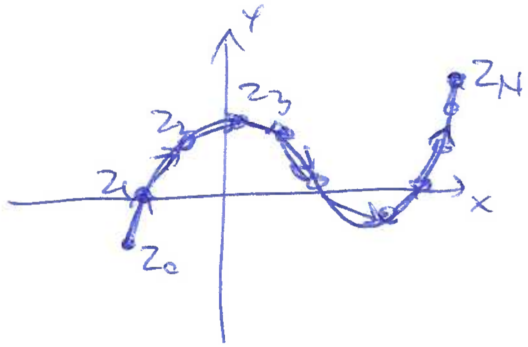
$\sin : D' \longrightarrow E'$

$\text{Arcsin} : E' \longrightarrow D'$ inverse fcn.

<u>Formula</u>: $\text{Arcsin}(w) = -i \,\text{Log}\left(iw + \sqrt{1-w^2}\right)$

Note: $\sqrt{1-w^2}$ can be chosen continuously on E'.

1.6 Line Integrals



γ oriented curve in $\cancel{\mathbb{R}^2}$. $\mathbb{C}^2 = \mathbb{R}^2$

$f : D \longrightarrow \mathbb{R}$ cont. fcn, $\gamma \subseteq D$.

Def $\displaystyle\int_\gamma f(z)\,dz$ = limit of $\displaystyle\sum_{i=1}^{N} f(z_i)\,(z_i - z_{i-1})$

= limit of $\sum f(z) \cdot \Delta z$

Example $\displaystyle\int_\gamma 1\,dz = z_N - z_0$

Parametrized curve

$\gamma(t) = x(t) + i\,y(t)$.

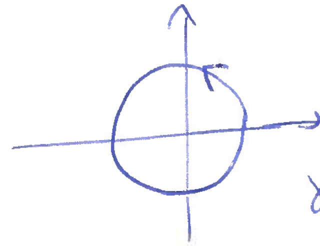
$\gamma'(t) = x'(t) + i\,y'(t)$.

$\gamma : (a, b) \longrightarrow \mathbb{C}$.

$$\int_\gamma f(z)\,dz = \int_{t=a}^{b} f(\gamma(t))\,\gamma'(t)\,dt.$$

$$= \text{limit of } \sum_{i=1}^{N} f(\gamma(t_i)) \cdot (\gamma(t_i) - \gamma(t_{i-1}))$$

Example $\gamma =$ unit circle



$\gamma(t) = e^{it}$

$t \in [0, 2\pi]$

$$\int_\gamma \mathrm{Re}(z)\,dz =$$

$$= \int_{t=0}^{2\pi} \mathrm{Re}(e^{it})\,\gamma'(t)\,dt$$

$\gamma(t) = \cos t + i \sin t$

$\gamma'(t) = -\sin(t) + i\cos(t)$

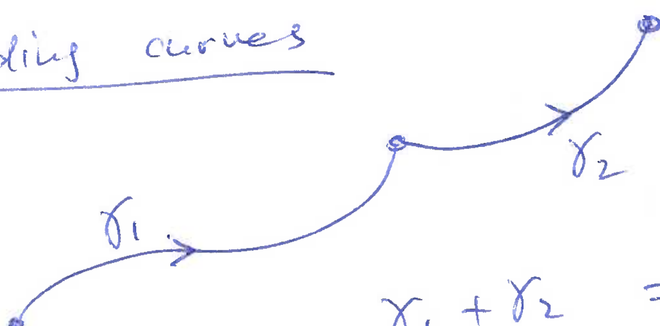
$$= \int_{t=0}^{2\pi} \cos(t)\left(-\sin(t) + i\cos(t)\right) dt$$

$$= -\int_{t=0}^{2\pi} \cos(t)\sin(t)\,dt + i\int_0^{2\pi} \cos(t)^2\,dt$$

$$\boxed{\cos(t)^2 = \frac{1}{2} + \frac{1}{2}\cos(2t)}$$

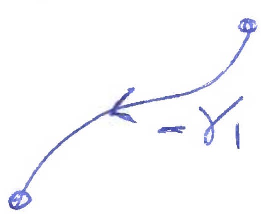
$$= i\pi$$

Adding curves



$\gamma_1 + \gamma_2 =$ combined curve.

$-\gamma_1 =$



~~Adding~~

$\gamma_1 : [a,b] \longrightarrow \mathbb{C}$

$\gamma_2 : [c,d] \longrightarrow \mathbb{C}$.

$(\gamma_1 + \gamma_2) : [a, b + (d-c)] \to \mathbb{C}$

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & a \le t \le b \\ \gamma_2(t - b + c) & b \le t \le b+d \end{cases}$$

Note: $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$

$$\int_{-\gamma_1} f(z)\,dz = -\int_{\gamma_1} f(z)\,dz.$$

Prop $\displaystyle\left|\int_a^b g(t)\,dt\right| \leq \int_a^b |g(t)|\,dt$

$$g: [a,b] \longrightarrow \mathbb{C}.$$

Consequence:

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right|$$

$$\leq \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt$$

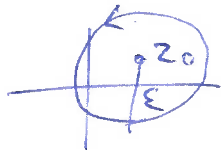
$$\leq \int_a^b M\cdot|\gamma'(t)|\,dt \qquad M = \max_{z\in\gamma}|f(z)|$$

$$= M\cdot \text{length}(\gamma).$$

Prop f cont. fcn. on D. $z_0 \in D$.

$B(z_0, r) \subseteq D$.

γ_ε :



$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0} \, dz = f(z_0)$$

Pf $\gamma_\varepsilon(t) = z_0 + \varepsilon e^{it}$, $0 \le t \le 2\pi$ $\quad \boxed{\gamma_\varepsilon'(t) = i\varepsilon e^{it}}$

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{it})}{\varepsilon e^{it}} \, i\varepsilon e^{it} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \varepsilon e^{it}) \, dt$$

$$\left| \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0} \, dz - f(z_0) \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \left(f(z_0 + \varepsilon e^{it}) - f(z_0) \right) \right|$$

$$\le \max_{0 \le t \le 2\pi} \left| f(z_0 + \varepsilon e^{it}) - f(z_0) \right|$$

$$\longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

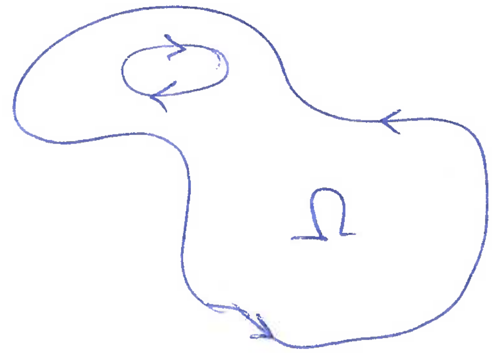
$\square$

2/6/2018 Complex

$\Omega \subseteq \mathbb{C}$ open connected.

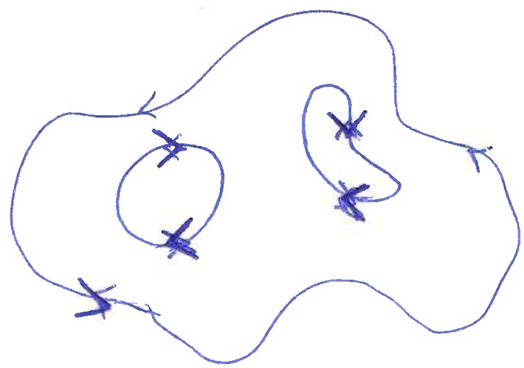
boundary = closed curve Γ

orient positively: Ω on your LEFT!



$\Gamma = \partial\Omega$

Example



$f: D \longrightarrow \mathbb{C}$ complex function.

$f(z) = p(z) + i\,q(z)$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(p(x+iy) + i\,q(x+iy)\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(p(x+iy) + i\,q(x+iy)\right)$$

Example

$$f(z) = \exp(z^2) = \exp(x^2-y^2+2ixy)$$

$$= e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right)$$

$$p(x+iy) = e^{x^2-y^2}\cos(2xy)$$

$$q(x+iy) = e^{x^2-y^2}\sin(2xy)$$

$$\frac{\partial f}{\partial x} = \frac{\partial p(x+iy)}{\partial x} + i\frac{\partial q}{\partial x}$$

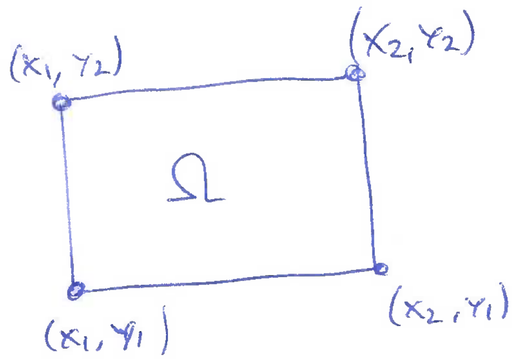
$$\frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + i\frac{\partial q}{\partial y}.$$

Green's Thm

Assume f has cont partial derivatives on open set D containing Ω and Γ. Then

$$\int_\Gamma f(z)\,dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$$

RECTANGLE



$$f(z) = p(z) + i\,q(z)$$

$$f(x+iy) = p(x+iy) + i\,q(x+iy)$$

$$f(x,y) = p(x,y) + i\,q(x,y) = (p(x,y),\ q(x,y))$$

$$\frac{\partial f}{\partial x} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} \qquad\qquad \frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + i\frac{\partial q}{\partial y}$$

$$i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy = i\iint_\Omega \left(\frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} + i\frac{\partial p}{\partial y} - \frac{\partial q}{\partial y}\right) dx\,dy$$

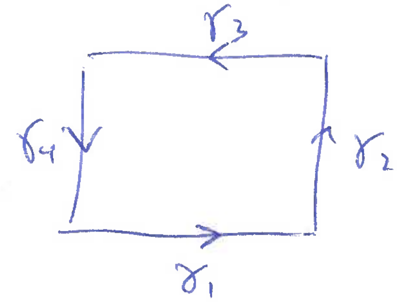
$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} i\frac{\partial p}{\partial x}\,dx\,dy - \int_{y_1}^{y_2}\int_{x_1}^{x_2} \frac{\partial q}{\partial x}\,dx\,dy - \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{\partial p}{\partial y}\,dy\,dx - \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{\partial q}{\partial y}\,d$$

$$= i\int_{y_1}^{y_2} \big(p(x_2+iy) - p(x_1+iy)\big)\,dy - i\int_{x_1}^{x_2}\big(q(x+iy_2) - q(x+iy_1)\big)\,dx$$

$$- \int_{y_1}^{y_2}\big(q(x_2+iy) - q(x_1+iy)\big)\,dy - \int_{x_1}^{x_2}\big(p(x+iy_2) - p(x+iy_1)\big)\,dx$$

3

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$



$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$\gamma_1(t) = t + iy_1$

$x_1 \le t \le x_2$

$\gamma_1'(t) = 1$

$$\int_{\gamma_1} f(z)\,dz = \int_{x_1}^{x_2} f(\gamma_1(t))\,\gamma_1'(t)\,dt$$

$$= \int_{x_1}^{x_2} \left(p(t+iy_1)\,dt + i\,q(t+iy_1)\,dt\right)$$

$$= \int_{x_1}^{x_2} p(x+iy)\,dx + i\int_{x_1}^{x_2} q(x+iy_1)\,dx$$

$\gamma_2(t) = x_2 + it$

$y_1 \le t \le y_2$

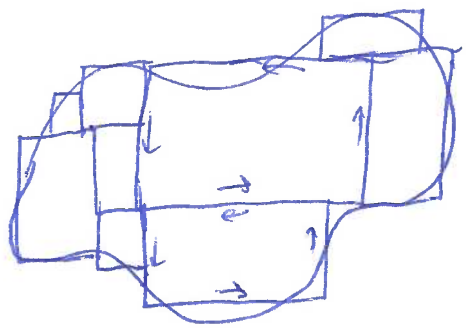
$\gamma_2'(t) = i$

$$\int_{\gamma_2} f(z)\,dz = \int_{y_1}^{y_2} f(x_2+it)\,i\,dt$$

$$= i\int_{y_1}^{y_2} p(x_2+iy)\,dy - \int_{y_1}^{y_2} q(x_2+iy)\,dy$$

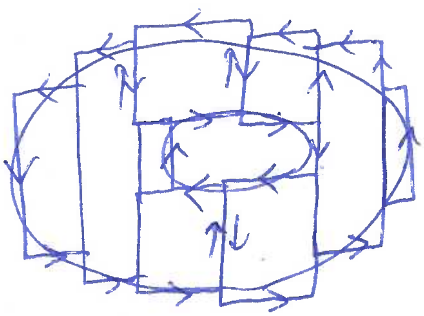
etc.

Any Region: Approximate with rectangles:



Note: Integrals along common sides cancel!

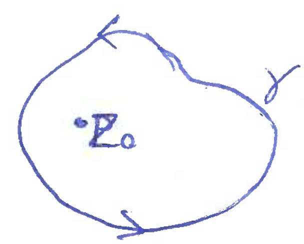
Holes:



IMPORTANT EXAMPLE:

γ pos. oriented simple curve, $z_0 \in \mathbb{C}$, $z_0 \notin \gamma$.

$$\int_\gamma \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ inside } \gamma \\ 0 & \text{if } z_0 \text{ outside } \gamma \end{cases}$$



Pf

$f(z) = \frac{1}{z-z_0}$ is defined on $D = \mathbb{C} \setminus \{z_0\}$.

$$f(z) = \frac{1}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{|z-z_0|^2} = p(x+iy) + i\,q(x+iy)$$

$$p = \frac{x-x_0}{|z-z_0|^2} \qquad q = \frac{y_0-y}{|z-z_0|^2} \qquad |z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\frac{\partial p}{\partial x} = \frac{1}{|z-z_0|^2} - 2\frac{x-x_0}{|z-z_0|^3}\,\frac{1}{2}|z-z_0|^{-1}\cdot 2(x-x_0)$$

$$= \frac{1}{|z-z_0|^2} - 2\frac{(x-x_0)^2}{|z-z_0|^4}$$

$$\cancel{\frac{\partial p}{\partial y} = -2\frac{x-x_0}{|z-z_0|^3}\,\frac{1}{2}|z-z_0|^{-1}\cdot 2(y-y_0) = -2\frac{(x-x_0)(y-y_0)}{|z-z_0|^4}}$$

$$\frac{\partial q}{\partial x} = -2\frac{y_0-y}{|z-z_0|^3}\,\frac{1}{2}|z-z_0|^{-1}\cdot 2(x-x_0) = 2\frac{(y-y_0)(x-x_0)}{|z-z_0|^4}$$

$$\cancel{\frac{\partial q}{\partial y} = -\frac{1}{|z-z_0|^2} - 2\frac{y_0-y}{|z-z_0|^3}\,\frac{1}{2}|z-z_0|^{-1}\cdot 2(y-y_0)}$$

$$\cancel{= -\frac{1}{|z-z_0|^2} + 2\frac{(y-y_0)^2}{|z-z_0|^4}}$$

$$\cancel{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} + i\frac{\partial q}{\partial x} + i^2\frac{\partial q}{\partial y}}$$

$$\cancel{\frac{\partial f}{\partial x} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{1}{|z-z_0|^2} + 2\frac{-(x-x_0)^2 + i\,[\text{illegible}](y-y_0)^2}{|z-z_0|^4}}$$

$$\cancel{|z-z_0|^2 \quad 2}$$

$=$

$$\frac{\partial f}{\partial x} = \frac{\partial p}{\partial x} + i\,\frac{\partial q}{\partial y}$$

$$= \frac{1}{|z-z_0|^2} + 2\,\frac{-(x-x_0)^2 + i(y-y_0)(x-x_0)}{|z-z_0|^4}$$

$$= \frac{|z-z_0|^2 - 2(x-x_0)(x-x_0 - iy + iy_0)}{|z-z_0|^4}$$

$$= \frac{|z-z_0|^2 - 2(x-x_0)(\bar{z}-\bar{z}_0)}{|z-z_0|^4}$$

$$= \frac{(z-z_0)(\bar{z}-\bar{z}_0) - 2(x-x_0)(\bar{z}-\bar{z}_0)}{|z-z_0|^4}$$

$$= \frac{(z-z_0 - 2x + 2x_0)(\bar{z}-\bar{z}_0)}{|z-z_0|^4}$$

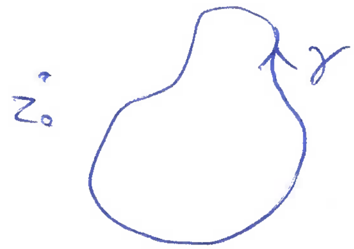
$$= \frac{-(\bar{z}-\bar{z}_0)(\bar{z}-\bar{z}_0)}{(z-z_0)^2(\bar{z}-\bar{z}_0)^2}$$

$$= -\frac{1}{(z-z_0)^2}$$

Similarly: $\qquad \dfrac{\partial f}{\partial y} = \dfrac{-i}{(z-z_0)^2}$

So $\quad \dfrac{\partial f}{\partial x} + i\,\dfrac{\partial f}{\partial y} = -\dfrac{1}{(z-z_0)^2} + i\,\dfrac{-i}{(z-z_0)^2} = 0.$

z_0 outside γ:



Greene's Thm:

$$\int_\gamma \frac{1}{z-z_0}\,dz = i \iint_{\not\ni \Omega} \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$$

$$= 0.$$

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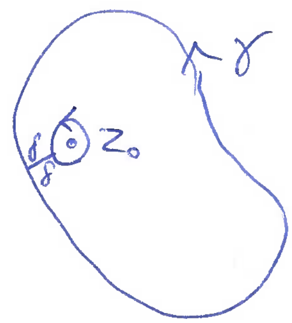
z_0 outside γ:



$$\gamma_\varepsilon(t) = z_0 + \varepsilon e^{it}$$
$$0 \le t \le 2\pi$$

Same arg: $\displaystyle\int_{\gamma+\delta-z_0-\delta} f(z)\,dz = 0.$

So $\displaystyle\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\gamma_\varepsilon} \frac{1}{z-z_0}\,dz$ $\quad \boxed{\gamma_\varepsilon'(t) = i\varepsilon e^{it}}$

$$= \int_0^{2\pi} \frac{1}{\gamma_\varepsilon(t) - z_0}\,\gamma_\varepsilon'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Complex Variables 2/8/2018

$D \subseteq \mathbb{C}$ open, $f : D \longrightarrow \mathbb{C}$ function, $z_0 \in D$.

<u>Def</u> f is <u>diff</u> at z_0, with <u>derivative</u> $f'(z_0)$, if

$$\frac{f(z) - f(z_0)}{z - z_0} \longrightarrow f'(z_0) \quad \text{as} \quad z \to z_0.$$

ANY DIRECTION! $f'(z_0) = \lim\limits_{\substack{h \to 0 \\ \text{in } \mathbb{C}}} \dfrac{f(z_0+h) - f(z_0)}{h}$

<u>Examples</u>

$f(z) = z^m$. $(f+g)'$ $(fg)'$ $\left(\frac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

$$\frac{d}{dz} g(f(z)) = g'(f(z))\, f'(z).$$

$$\frac{d}{dz} e^z = e^z \qquad \exp'(z) = \exp(z).$$

$$\frac{(z+h)^m - z^m}{h} = \frac{\sum_{i=1}^{m} \binom{m}{i} z^{m-i} h^i}{h} \to m z^{m-1}$$

$$\frac{g(f(z_0+h)) - g(f(z_0))}{h} = \frac{g(f(z_0+h)) - g(f(z_0))}{f(z_0+h) - f(z_0)} \cdot \frac{f(z_0+h) - f(z_0)}{h}$$

$$\longrightarrow g'(f(z_0))\, f'(z_0)$$

(NOT rigorous!)

<u>Non-examples</u>

$$f(z) = \overline{z}$$

$$f(z) = \mathrm{Re}(z).$$

$$f(z) = |z|$$

$$f(z) = |z|^2 = x^2 + y^2.$$

$$\left|\frac{e^h-1}{h}-1\right| = \left|\frac{e^h-1-h}{h}\right| \qquad h = s+it$$

$$= \left|\frac{e^s(\cos t + i\sin t) - 1 - s - it}{s+it}\right|$$

$$\leq \frac{|e^s\cos(t)-1-s|}{|s+it|} + \frac{|e^s\sin(t)-t|}{|s+it|}$$

$$\leq e^s\frac{|\cos(t)-1|}{|s+it|} + \frac{|e^s-1-s|}{|s+it|} + e^s\frac{|\sin(t)-t|}{|s+it|}$$

$$+ \frac{|e^s t - t|}{|s+it|}$$

$$\leq e^s\frac{|\cos(t)-1|}{|t|} + \frac{|e^s-1-s|}{|s|} + e^s\frac{|\sin(t)-t|}{|t|}$$

$$+ (e^s-1)$$

$$\therefore \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

$f = u + iv$. analytic ③

$$f'(z_0) = \lim \left[\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i \, \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \right]$$

$$= \frac{\partial u}{\partial x}(z_0) + i \, \frac{\partial v}{\partial x}(z_0)$$

$$f'(z_0) = \lim \left(\frac{u(x_0, y_0+k) - u(x_0, y_0)}{ik} + i \, \frac{v(x_0, y_0+k) - v(z_0)}{ik} \right)$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \, \frac{\partial v}{\partial y} \right)$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \boxed{\begin{aligned} -\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned}}$$

Cauchy Riemann

$f = u + iv$ analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

Harmonic

$u: D \longrightarrow \mathbb{R}$, $\quad D \subseteq \mathbb{C}$ open.

Laplacian $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}$

Harmonic $\quad \Delta u = 0$ on D.

Assume $f: D \longrightarrow \mathbb{C}$ analytic,
$f = u + iv$.

Then u and v are Harmonic on D

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0.$$

(Assume cont deriv. of 2nd order)

Harmonic Conjugate:

Assume $u = \mathrm{Re}(f)$ known, f analytic.
D connected open.

$$\left.\begin{aligned} \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned}\right\}$$ determines $v(z)$ up to constant on D.

~~Then~~ v is Harm conjugate of u if $f = u + iv$ analytic.

Thm $f : D \longrightarrow \mathbb{C}$ analytic, D connected.

If $|f|$ constant or $\mathrm{Re}(f)$ constant,

then $f(z)$ constant.

Pf

Assume $u = \mathrm{Re}(f)$ constant

$v = \mathrm{Im}(f)$.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0 \quad \Bigg| \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0.$$

$\therefore$ v constant.

Assume $|f|$ constant. (and $|f| \neq 0$).

Then $\overline{f(z)} = \dfrac{\overline{f(z)}\, f(z)}{f(z)} = \dfrac{|f(z)|^2}{f(z)}$ analytic.

$\Rightarrow \quad i\left(f(z) + \overline{f(z)}\right) = 2i\,\mathrm{Re}(f(z))$ Analytic.

Real part $= 0$ constant

$\Rightarrow 2i\,\mathrm{Re}(f(z))$ constant

$\Rightarrow f(z)$ constant.

$\square$

Thm Assume $f = u + iv$, $z_0 \in D$, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ def. + cont. on $B(z_0, r) \subseteq D$.

Assume CR eqns hold at z_0:

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \& \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).$$

Then f diff at z_0.

Pf Taylor expansions: $z_0 = x_0 + iy_0$

$$u(x_0 + s, y_0 + t) = u(x_0, y_0) + s\frac{\partial u}{\partial x}(z_0) + t\frac{\partial v}{\partial y}(z_0) + sE_1(s,t) + tE_2(s,t)$$

$$v(x_0+s, y_0+t) = v(z_0) + s\frac{\partial v}{\partial x}(z_0) + t\frac{\partial v}{\partial y}(z_0) + sE_3(s,t) + tE_4(s,t)$$

where $E_i(s,t) \to 0$ as $(s,t) \to (0,0)$.

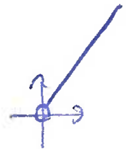
$h = s + it$.

$$\boxed{\cancel{\frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{h}\left(s\left(\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)\right) + it\left(\frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right)\right)}}$$

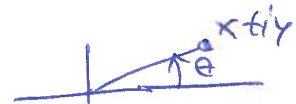
$$f(z_0+h) = f(z_0) + \left(\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)\right)\cdot h + s(E_1 + iE_3) + t(E_2 + iE_4)$$

$$\frac{f(z_0+h) - f(z_0)}{h} \to \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) \text{ as } h \to 0.$$

$\square$

<u>Example</u> $D = \mathbb{C} - \text{ray}$ 

$\log(z)$ diff. on D. analytic on D.



$\tan\theta = \frac{y}{x}$

<u>PP</u> $\log(z) = \ln|z| + i \arg(z)$

$$= \frac{1}{2}\ln(x^2+y^2) + i \arctan(y/x)$$

$u = \frac{1}{2}\ln(x^2+y^2)$ $\qquad$ $v = \arctan(y/x)$.

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

Check: CR eqns hold. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\log'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x-iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}.$$

$\square$

<u>Example</u> $f(z) = z^\alpha$, $\alpha \in \mathbb{C}$. $\qquad f(z) = \exp(\alpha \log(z))$

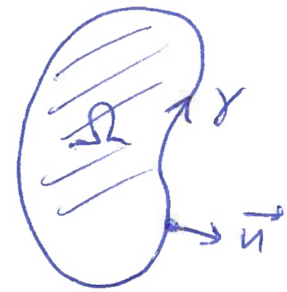
$$f'(z) = \exp(\alpha\log(z))\cdot\frac{\alpha}{z} = z^\alpha\cdot\frac{\alpha}{z} = \alpha z^{\alpha-1}$$

Complex Variables 2/8/2018

1) <u>Green's Thm</u> $\vec{F} = (u, v)$ vector field.

$\gamma(t) = (x(t), y(t))$ closed curve.

$a \le t \le b$.



<u>Curl form:</u>

$$\int_\gamma \vec{F} \cdot \gamma'(t)\, dt = \iint_\Omega \text{curl}(F)\, dx\, dy$$

$$\int_a^b \Big(u(\gamma(t))\, x'(t) + v(\gamma(t))\, y'(t) \Big)\, dt = \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

<u>Divergence thm:</u>

$$\int_\gamma \vec{F} \cdot \vec{n}\, ds = \iint_\Omega \text{div}(\vec{F})\, dx\, dy$$

$$\int_a^b \Big(u(\gamma(t))\, y'(t) - v(\gamma(t))\, x'(t) \Big)\, dt = \iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\, dy$$

<u>Def</u> $f(z) = \bar{F} = u(z) + i\, v(z)$

$$\int_\gamma f(z)\, dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

$$\int_a^b \Big(u(\gamma(t)) + i\, v(\gamma(t)) \Big) \big(x'(t) + i\, y'(t) \big) = \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + i \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\, dy$$

Re = Re : Curl Thm

Im = Im : Div. Thm.

2) γ pos orient simple curve

$z_0 \in \mathbb{C}$, $z_0 \notin \gamma$.

$$\int_\gamma \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ inside } \gamma \\ 0 & \text{else} \end{cases}$$

3) Diff. $f : D \to \mathbb{C}$.

Examples + non-examples.

4) Cauchy-Riemann eqns

$f = u + iv$ analytic $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

5) $f = u + iv$ analytic $\Rightarrow$ u and v harmonic.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

6) Harmonic conjugate. skip?

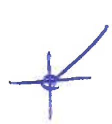
7) Thm $f : D \to \mathbb{C}$ analytic, D connected.

$|f|$ const or $\mathrm{Re}(f)$ const $\Rightarrow$ f const.

8) Thm $f : D \to \mathbb{C}$, $f = u + iv$, $z_0 \in D$.

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ def + cont on $B(z_0, r) \subseteq D$.

CR at $z_0 \Rightarrow f$ diff at z_0.

9) $D = \mathbb{C} - \text{ray}$  $\log(z)$ analytic on z, $\log'(z) = \frac{1}{z}$.

$$f(z) = u + iv$$

$$i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

$$= \operatorname{curl}\left(\overline{f(z)}\right) + i \operatorname{div}\left(\overline{f(z)}\right)$$

Cauchy-Riemann $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Updownarrow$

$$i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

$\Updownarrow$

Vector field $\overline{f(z)}$ is locally conservative and curl-free.

$$\int_\Gamma f(z)\,dz = \int_\Gamma (u+iv)(dx+idy) = \int_\Gamma (u\,dx - v\,dy) + i\int_\Gamma (u\,dy + v\,dx)$$

$$= \int_\Gamma \overline{f(z)} \cdot \vec{\tau}\,ds + i\int_\Gamma \overline{f(z)} \cdot \vec{n}\,ds$$

Green's Thm $\Vert$

$$\iint_\Omega i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dx\,dy = \iint_\Omega \operatorname{curl}\left(\overline{f(z)}\right)dx\,dy + i\iint_\Omega \operatorname{div}\left(\overline{f(z)}\right)dx\,dy$$

Complex Variables 2/13/2018

Midterm 1: Tue 2/20 in class

2.2 Power Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$ power ser. with center $z_0 \in \mathbb{C}$, coefs $a_n \in \mathbb{C}$.

E.g. $f(z) = \frac{1}{z-z_0} = \sum_{n=0}^{\infty} (z-z_0)^n$ for $|z-z_0| < 1$.

Convergence?

Assume $r > 0$ is such that seq. $(a_n r^n)_{n \in \mathbb{N}}$ is bounded.
($\exists M > 0$ s.t. $|a_n| r^n \le M$ for all n.)

Then $\sum a_n (z-z_0)^n$ is abs. conv. for $z \in B(z_0, r)$

Since $$\sum_0^{\infty} |a_n| \, |z-z_0|^n = \sum_0^{\infty} |a_n| r^n \left|\frac{z-z_0}{r}\right|^n \le \sum_0^{\infty} M \left|\frac{z-z_0}{r}\right|^n$$

Assume $(a_n r^n)_{n \in \mathbb{N}}$ is NOT bounded.
Then $\sum a_n (z-z_0)^n$ diverges for $z \notin B(z_0, r)$

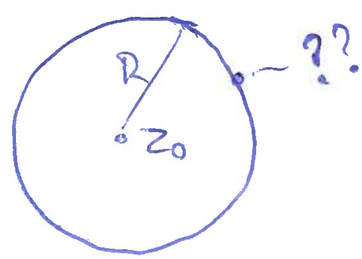
since $|a_n| \, |z-z_0|^n \ge |a_n| r^n$ not bounded, so $\not\to 0$.

Radius of convergence R

$\exists ! \; R \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ such that

$(a_n r^n)_n$ bounded for $0 \le r < R$ and

$(a_n r^n)_n$ unbounded for $R < r$



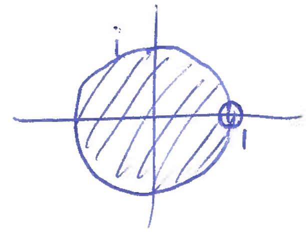
$\sum a_n (z-z_0)^n$ abs. conv. for $z \in B(z_0, R)$

diverges for $|z-z_0| > R$

Example $f(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n$

Converges for $|z|<1$, diverges for $|z|>1$.

$R = 1$.



Note: $f(1) = \sum \frac{1}{n}$ diverges

$f(-1) = \sum (-1)^n \frac{1}{n}$ converges.

In fact: $f(z)$ converges $\Leftrightarrow$ $|z| \le 1$ and $z \ne 1$.

Thm $f(z) = \sum a_n (z-z_0)^n$, $R > 0$. (pos or ∞).

(a) If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists then

$$1/R = \lim_{n\to\infty} |a_{n+1}/a_n|$$

(b) If $\lim \sqrt[n]{|a_n|}$ exists then

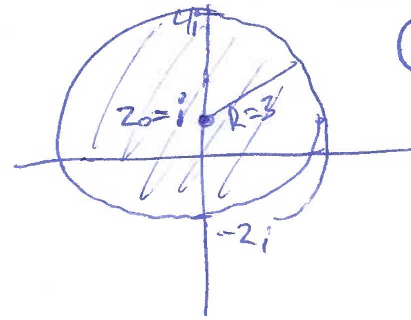
$$1/R = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Pf (b) $L = \lim \sqrt[n]{|a_n|}$

$$\lim \sqrt[n]{|a_n z^n|} = \lim \sqrt[n]{|a_n|} \cdot |z| = L \cdot |z|$$

$$\left.\begin{array}{ll} < 1 & \text{for } |z| < \frac{1}{L} \\ > 1 & \text{for } |z| > \frac{1}{L} \end{array}\right\} \Rightarrow R = \frac{1}{L}.$$

$\square$



Example $\quad f(z) = \sum_{n=1}^{\infty} \frac{n^2}{(i3)^n} (z-i)^n$

$z_0 = i \quad , \quad a_n = \frac{n^2}{(3i)^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2\, 3^{-n-1}}{n^2\, 3^{-n}} \longrightarrow \frac{1}{3}$$

$R = 3$

$$\sqrt[n]{|a_n|} = \frac{1}{3}\left(\sqrt[n]{n}\right)^2 \longrightarrow \frac{1}{3}$$

Example $\quad g(z) = \sum \frac{n^2}{(3i)^n} z^{5n} = \frac{1}{3i} z^5 + \frac{4}{-9} z^{10} + \cdots$

converges if $|z^5| < 3$

$R = \sqrt[5]{3}.$

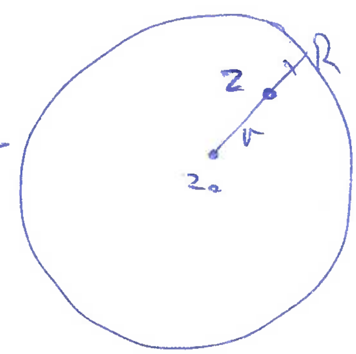
Derivative of Power Series

Thm Assume $f(z) = \sum_{0}^{\infty} a_n (z-z_0)^n$ has $R > 0$.

Then $\sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$ converges on $B(z_0, R)$.

Pf $\quad z \in B(z_0, R).$

$r = |z - z_0| < R. \qquad s = \frac{r+R}{2}.$



$$\sum \left| n a_n (z-z_0)^{n-1} \right| = \sum \left| n a_n r^{n-1} \right|$$

$$= \sum \left| n \frac{r^{n-1}}{s^n} \right| \cdot \left| a_n s^n \right|$$

abs conv. since $\frac{r}{s} < 1$ and $s < R$.

$\square$

<u>Thm</u> $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $R>0$

Then $f(z)$ analytic on $B(z_0, R)$,

with $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$

<u>Pf</u>

Let $z \in B(z_0, R)$. $w = z - z_0$. $|w| < R$.

$$f(z+h) - f(z) = \sum \left(a_n (w+h)^n - a_n w^n\right)$$

Set $g(z) = \sum_{1}^{\infty} n a_n (z-z_0)^{n-1}$.

$$\frac{f(z+h)-f(z)}{h} - g(z) = \sum_{n=1}^{\infty} a_n \left(\frac{(w+h)^n - w^n}{h} - n w^{n-1} \right)$$

$$\frac{(w+h)^n - w^n}{h} - n w^{n-1} = \frac{\sum_{j=1}^{n} \binom{n}{j} w^{n-j} h^j}{h} - n w^{n-1}$$

$$= \sum_{j=2}^{n} \binom{n}{j} w^{n-j} h^{j-1}$$

$$\left| \frac{(w+h)^n - w^n}{h} - n w^{n-1} \right| \le \sum_{j=2}^{n} \binom{n}{j} |w|^{n-j} |h|^{j-1}$$

$\delta = \frac{1}{2}(R - |w|)$ Assume $|h| < \delta$.

$$\le |h| \sum_{j=2}^{n} \binom{n}{j} (R-2\delta) \delta^{j-2}$$

$$= |h|\, \delta^{-2} \sum_{j=2}^{n} \binom{n}{j} (R-2\delta)^{n-j} \delta^{j}$$ ⑤

$$< |h|\, \delta^{-2} \sum_{j=0}^{n} \binom{n}{j} (R-2\delta)^{n-j} \delta^{j}$$

$$= |h|\, \delta^{-2} \big((R-2\delta) + \delta \big)^{n}$$

$$= |h|\, \delta^{-2} (R-\delta)^{n}$$

$$\left| \frac{f(z+h)-f(z)}{h} - g(z) \right| \le \underbrace{|h|}_{\to 0.}\, \delta^{-2} \underbrace{\sum_{n=2}^{\infty} |a_n| (R-\delta)^{n}}_{\text{convergent}}$$

$\therefore \quad \dfrac{f(z+h)-f(z)}{h} \longrightarrow g(z), \quad$ as $h \to 0$

$\square$

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $R > 0$.

$f(z)$ is ∞ diff on $B(z_0, R)$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)(z-z_0)^{n-k}$$

AND $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

Example $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$a_n = \frac{1}{n!}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{1}{n+1} \to 0$. $\qquad R = \infty$.

$$f'(z) = \sum_{n=1}^{\infty} n \frac{1}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = f(z).$$

$f(0) = 1$.

Claim: $f(z) = e^z$.

$\left(f(z)\, e^{-z}\right)' = 0$. for all $z \in \mathbb{C}$.

$\Rightarrow f(z)\, e^{-z}$ constant.

$f(0)\, e^{-0} = 1$.

$f(z) = e^z$.

Ex $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$

Thm

$f(z) = \sum a_n (z-z_0)^n$, $g(z) = \sum b_n (z-z_0)^n$ on $B(z_0, R)$

Then $\underbrace{f(z)g(z)}_{(fg)(z)=} = \sum c_n (z-z_0)$ for $z \in B(z_0, R)$

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

Example

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad |z| < 1.$$

Expand $\frac{1}{1-z} e^z$ at $z_0 = 0$.

$$\frac{1}{1-z} e^z = (1+z+z^2+\cdots)(1+z+\tfrac{1}{2}z^2+\tfrac{1}{6}z^3+\cdots)$$

$$= 1 + 2z + \tfrac{5}{2}z^2 + \cdots$$