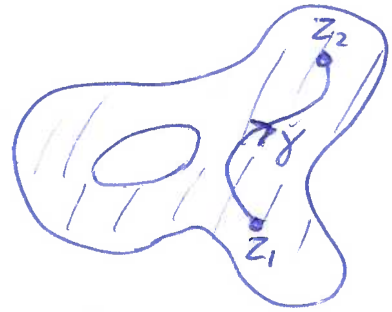
Complex Variables 2/15/2018

MT1 2/20

2.3 Cauchy's Thm & Formula

$D \subseteq \mathbb{C}$ open, $f : D \longrightarrow \mathbb{C}$ analytic.

$\gamma \subseteq D$ oriented curve,
from z_1 to z_2.



$\gamma : [a,b] \longrightarrow D$

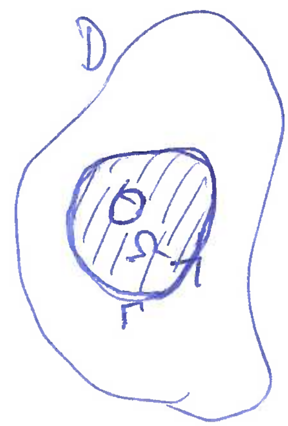
$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}\big(f(\gamma(t)) \big)\,dt$$
$$= f(\gamma(b)) - f(\gamma(a))$$
$$= f(z_2) - f(z_1)$$

Green's Thm $\quad f = u + iv : D \longrightarrow \mathbb{C}$

Assume $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist + continuous.

$\Omega \subseteq \mathbb{C}$ open, $\partial\Omega = \Gamma$ piecewise smooth curve,
positively oriented.

$\overline{\Omega} = \Omega \cup \Gamma \subseteq D$.



Then $\displaystyle\int_\Gamma f(z)\,dz = \iint_\Omega i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$

$$i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$
$$= \operatorname{curl}\left(\overline{f(z)}\right) + i \operatorname{div}\left(\overline{f(z)}\right)$$

$\overline{f(z)} = u - iv$
vector field

Cauchy–Riemann $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\Updownarrow$$

$$i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

$$\Updownarrow$$

fluxless.

$\overline{f(z)}$ is locally conservative and ~~free~~ free.

Fact: Every analytic function $f: D \longrightarrow \mathbb{C}$ has cont. partial derivatives.

Equivalent: $f': D \longrightarrow \mathbb{C}$ is continuous

since $\frac{\partial f}{\partial x} = f'(z)$ and $\frac{\partial f}{\partial y} = i f'(z)$

$\boxed{\text{Assume without proof}}$

$$\left(\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x+iy) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} i \, \frac{f(x+iy+ih)}{ih} \right)$$

Cauchy's Thm

$f: D \longrightarrow \mathbb{C}$ analytic, $D \subseteq \mathbb{C}$ open.

$\gamma \subseteq D$ simple closed curve, s.t. inside $\Omega \subseteq D$.

Then $\displaystyle\int_\gamma f(z)\,dz = 0$

Proof Green $\Rightarrow \displaystyle\int_\gamma f(z)\,dz = \iint_\Omega i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$

$= 0$

□

Def $D \subseteq \mathbb{C}$ domain (open & connected).

Simply connected: If $\gamma \subseteq D$ any simple closed curve, then (inside of γ) $\subseteq D$.

Equiv: No "holes" in D.

Examples

- $\mathbb{C}$. • $B(0,1)$.
- $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

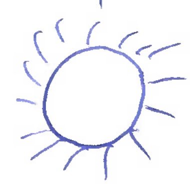
- $\{z \in \mathbb{C} \mid 1 < |z| < 2,\ z \notin \mathbb{R}_+\}$

-  $B(0,1) - \mathbb{R}_{\geq 0}$

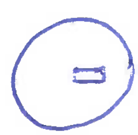
Non-examples

- $\mathbb{C} \setminus \{0\}$

- $\{z \in \mathbb{C} \mid |z| \geq 1\}$



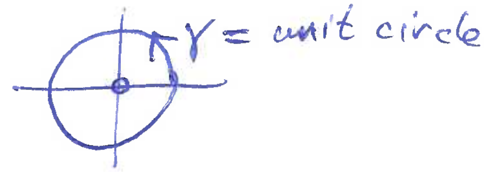
- $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$

-  $B(0,1) - [0, \tfrac{1}{2}]$.

Goal: $f: D \longrightarrow \mathbb{C}$ analytic, D simply connected,
$\gamma \subseteq D$ any closed curve. (piecewise smooth.)
Then $\int_\gamma f(z)\,dz = 0$.

Non-Example

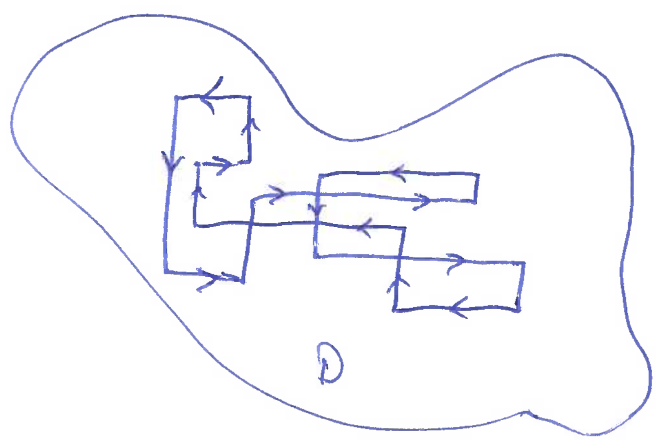
$$\int_{\text{unit circle}} \frac{1}{z}\,dz = 2\pi i$$



Reason: $f(z) = \frac{1}{z}$ is analytic on $D = \mathbb{C} \setminus \{0\}$,
D NOT simply connected!

Techical lemma

$f: D \longrightarrow \mathbb{C}$ analytic, D simply conn.
$\Gamma \subseteq D$ closed curve consisting of horiz + vert line segm.
Then $\int_\Gamma f(z)\,dz = 0$



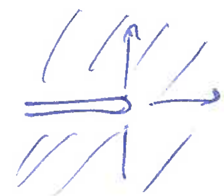
Idea:

$$\int_\Gamma f(z)\,dz = \sum_i \int_{\Gamma_i} f(z)\,dz$$

$\Gamma = \Gamma_1 + \dots + \Gamma_k$, Γ_i simple closed curve.

Thm $f: D \to \mathbb{C}$ analytic, $D \subseteq \mathbb{C}$ simply conn.
Then $\exists$ analytic $F: D \to \mathbb{C}$ with $F' = f$.

Non-Example
$f(z) = \frac{1}{z}$ is analytic on $D = \mathbb{C} \setminus \{0\}$.
$f(z)$ has no antiderivative on D.
(But $\frac{1}{z}$ has antiderivative on $\mathbb{C} - \mathbb{R}_{\leq 0}$)
$\text{Log}(z)$



Proof
Choose $z_0 \in D$ any point. "Set $F(z_0) = 0$."
Given $z \in D$, let $\gamma \subseteq D$ curve from z_0 to z, consisting of horiz/vert line segm.

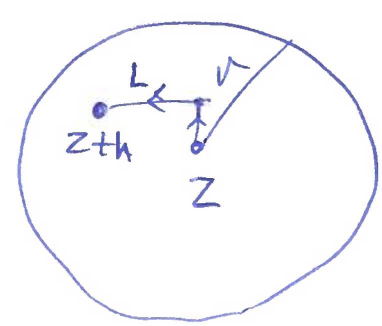
Def $F(z) = \int_\gamma f(\zeta)\,d\zeta$

Note: If γ_1 other such curve then Tech lemma $\Rightarrow \int_\gamma f(\zeta)\,d\zeta - \int_{\gamma_1} f(\zeta)\,d\zeta = 0$

$\therefore$ $F(z)$ well def.

Compute $F'(z)$:
Choose $r > 0$ s.t. $B(z, r) \subseteq D$. Let $h \in \mathbb{C}$, $|h| < r$.
Let L be "direct" horiz/vert curve from z to $z+h$

$F(z+h) = \int_\gamma f(\zeta)\,d\zeta + \int_L f(\zeta)\,d\zeta$



$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_L f(\zeta)\, d\zeta - f(z) \right|$$

$$= \left| \int_L \frac{f(\zeta) - f(z)}{h}\, d\zeta \right|$$

$$\leq \frac{1}{|h|} \,\text{length}(L) \left(\max_{\zeta \in L} |f(\zeta) - f(z)| \right)$$

$$\leq 2 \max_{\zeta \in L} |f(\zeta) - f(z)|$$

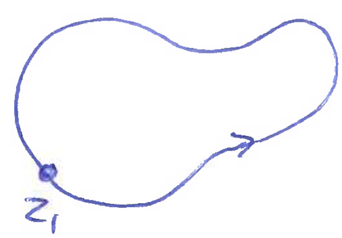
$$\longrightarrow 0 \quad \text{as } h \to 0.$$

$\square$

<u>Cor</u> $f : D \longrightarrow \mathbb{C}$ analytic, D simply conn.

$\gamma \subseteq D$ any closed curve (piecewise smooth).

Then $\displaystyle\int_\gamma f(z)\, dz = 0$



<u>Pf</u> let $F : D \longrightarrow \mathbb{C}$, $F' = f$

$$\int_\gamma f(z)\, dz = \int_\gamma F'(z)\, dz = F(z_1) - F(z_1) = 0$$

$\square$

Cauchy's formula

Thm $f : D \longrightarrow \mathbb{C}$ analytic, $D \subseteq \mathbb{C}$ open.

$\gamma \subseteq D$ simple closed curve, pos orient.

Assume (inside of γ) $\subseteq D$.

Then for every z_0 inside γ (Ω):

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$$

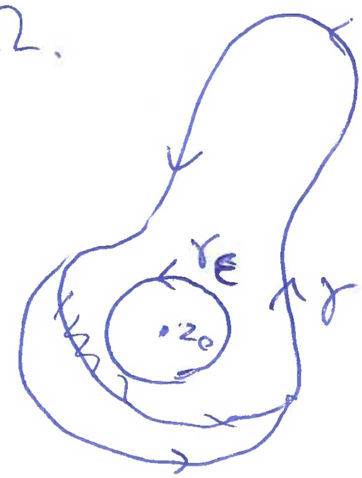
Proof

$$g(z) = \frac{f(z)}{z-z_0}$$

is analytic on

$D - \{z_0\}$

Choose $\varepsilon > 0$ s.t. $B(z_0, 2\varepsilon) \subseteq \Omega$.



Green's Thm $\Rightarrow$

$$\int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{z-z_0}\,dz$$

indep. of ε.
(since $g(z)$ analytic.) on $D - \{z_0\}$.

Example 10 in 1.6:

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,dz \longrightarrow f(z_0) \quad \text{as } \varepsilon \to 0.$$

$$\therefore \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0).$$

$\square$

Redo Ex 10: $\quad \gamma_\varepsilon(t) = z_0 + \varepsilon e^{it} \qquad 0 \le t \le 2\pi.$

$$\left| \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\, dz - f(z_0) \right|$$

$$= \left| \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(z_0 + \varepsilon e^{it})}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt - f(z_0) \right|$$

$$= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \varepsilon e^{it})\, dt - f(z_0) \right|$$

$$= \left| \frac{1}{2\pi} \int_0^{2\pi} \left(f(z_0 + \varepsilon e^{it}) - f(z_0) \right) dt \right|$$

$$\le \frac{1}{2\pi} \cdot 2\pi \cdot \max_{z \in \gamma_\varepsilon} |f(z) - f(z_0)|$$

$\to 0$ as $\varepsilon \to 0$. since f cont.

Cauchy's Formula

Thm $f: D \longrightarrow \mathbb{C}$ analytic, $D \subseteq \mathbb{C}$ open.

$\gamma \subseteq D$ simple closed curve, positively oriented.

Assume inside Ω of γ is $\subseteq D$.

Then for $z_0 \in \Omega$:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\, dz$$

Proof

$$g(z) = \frac{f(z)}{z-z_0} \quad \text{is analytic on } D - \{z_0\}$$

Choose $\varepsilon > 0$ s.t. $B(z_0, 2\varepsilon) \subseteq \Omega$.

Set $\Omega_\varepsilon = \{ z \in \Omega \mid |z| > \varepsilon \}$, $\quad \overline{\Omega_\varepsilon} \subseteq D - \{z_0\}$

Green's Thm $\Rightarrow$

$$0 = i \iint_{\Omega_\varepsilon} \left(\frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} \right) dx\,dy = \int_\gamma g(z)\,dz - \int_{\gamma_\varepsilon} g(z)\,dz$$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\, dz = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\, dz \quad \text{independent of } \varepsilon$$

Example 1.6.10: $f(z)$ continuous $\Rightarrow$

$$\frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\, dz \longrightarrow f(z_0) \quad \text{as} \quad \varepsilon \longrightarrow 0$$

$\square$

Ex 1.6.10:

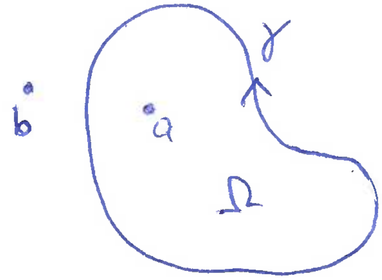
$$\left| \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\, dz - f(z_0) \right| = \left| \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z-z_0}\, dz \right|$$

$$\leq \frac{1}{2\pi} \operatorname{length}(\gamma_\varepsilon) \max_{z \in \gamma_\varepsilon} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \max_{z \in \gamma_\varepsilon} |f(z) - f(z_0)|$$

since $\operatorname{length}(\gamma_\varepsilon) = 2\pi\varepsilon$ and $|z - z_0| = \varepsilon$.

Example $\gamma \subseteq \mathbb{C}$ simple closed curve, pos. orient.

$a \neq b \in \mathbb{C} \setminus \gamma$. Find $\frac{1}{2\pi i}\int_\gamma \frac{e^z}{(z-a)(z-b)}dz$



$\boxed{A}$ a inside γ, b outside γ.

$f(z) = \frac{e^z}{z-b}$ analytic on $D \setminus \{b\}$

and $\overline{\Omega} \subseteq D \setminus \{b\}$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a) = \frac{e^a}{a-b}$$

$\boxed{B}$ b inside γ, a outside γ

$g(z) = \frac{e^z}{z-a}$ analytic on $D \setminus \{a\}$ and $\overline{\Omega} \subseteq D \setminus \{a\}$.

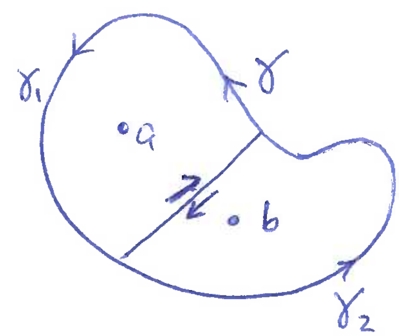
$$\frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-b}dz = g(b) = \frac{e^b}{b-a}$$

$\boxed{C}$ a, b both outside γ

$h(z) = \frac{e^z}{(z-a)(z-b)}$ analytic on $D \setminus \{a,b\} \supseteq \overline{\Omega}$.

Cauchy's Thm $\Rightarrow \frac{1}{2\pi i}\int_\gamma h(z)dz = 0$

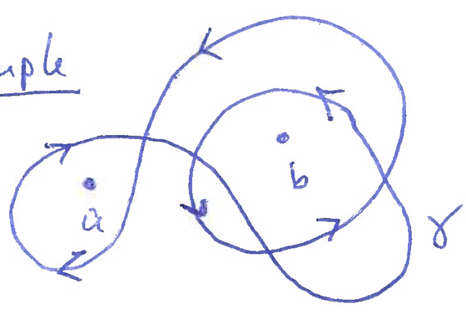
$\boxed{D}$ a, b both inside γ.



$$\frac{1}{2\pi i}\int_\gamma \frac{e^z}{(z-a)(z-b)}dz = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(z)}{z-a}dz + \frac{1}{2\pi i}\int_{\gamma_2}\frac{g(z)}{z-b}dz$$

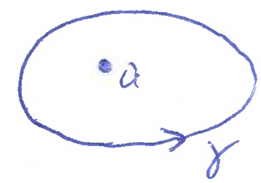
$$= \frac{e^a}{a-b} + \frac{e^b}{b-a}$$

Example



$$\frac{1}{2\pi i}\int_\gamma \frac{e^z}{(z-a)(z-b)} = 2\frac{e^a}{a-b} - \frac{e^b}{b-a}$$

<u>Example</u> a inside γ. $\frac{1}{2\pi i}\int_\gamma \frac{e^z}{(z-a)^2}\,dz = ?$



<u>Guess</u>: $\lim_{b\to a} \frac{1}{2\pi i}\int_\gamma \frac{e^z}{(z-a)(z-b)}\,dz =$

$$\lim_{b\to a}\left(\frac{e^a}{a-b}+\frac{e^b}{b-a}\right) = \lim_{b\to a}\frac{e^a-e^b}{a-b} = \exp'(a)$$

2.4 Consequences of Cauchy

<u>Main point</u>: $f: D \longrightarrow \mathbb{C}$ analytic, $z_0 \in D$, $B(z_0,R)\subseteq D$.

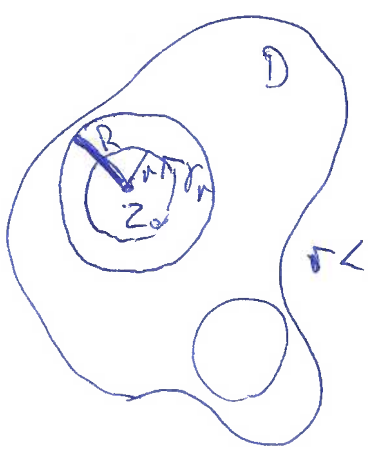
Then $f(z)$ is given by power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

valid for $z\in B(z_0,R)$, radius of conv. $\geq R$.

<u>Find a_m</u>: $$\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{(z-z_0)^{m+1}}\,dz = \frac{1}{2\pi i}\int_{\gamma_r}\sum_{n=0}^{\infty} a_n (z-z_0)^{n-m-1}\,dz$$

$$\underset{(!)}{=} \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i}\int_{\gamma_r}(z-z_0)^{n-m-1}\,dz$$



$r<R$

If $n\neq m$: $$\int_{\gamma_r}(z-z_0)^{n-m-1}\,dz = \int_{\gamma_r}\left(\frac{(z-z_0)^{n-m}}{n-m}\right)'dz = 0$$

If $n=m$: $$\int_{\gamma_r}(z-z_0)^{-1}\,dz = 2\pi i$$

$$\therefore \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{(z-z_0)^{m+1}}\,dz = a_m$$

Assuming that power series exist!!

Interchanging integral and sum

$$\int_{x=1}^{\infty}\left(\int_{y=1}^{\infty}\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\right)dx = \int_{x=1}^{\infty}\left[\frac{y}{x^2+y^2}\right]_{y=1}^{\infty}dx$$

$$= \int_{x=1}^{\infty}\left(0-\frac{1}{x^2+1}\right)dx = \Big[-\operatorname{atan}(x)\Big]_{x=1}^{\infty} = -\frac{\pi}{2}+\frac{\pi}{4}$$

$$= -\frac{\pi}{4}$$

$$\int_{y=1}^{\infty}\left(\int_{x=1}^{\infty}\frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right)dy = +\frac{\pi}{4}$$

"Reason":

$$\int_{y=1}^{\infty}\int_{x=1}^{\infty}\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|dx\,dy = 2\int_{y=1}^{\infty}\int_{x=y}^{\infty}\frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy$$

$$= 2\int_{y=1}^{\infty}\left[\frac{-x}{x^2+y^2}\right]_{x=1}^{\infty}dy = 2\int_{1}^{\infty}\frac{y}{y^2+y^2}\,dy$$

$$= \int_{1}^{\infty}\frac{1}{y}\,dy = +\infty.$$

<u>Thm</u> $\{f_n\}_{n\geq 0}$ seq. of cont. fcns. $f_n:[a,b]\longrightarrow\mathbb{R}$.

Then $\displaystyle\sum_{n=0}^{\infty}\int_a^b|f_n(t)|\,dt = \int_a^b\sum_{n=0}^{\infty}|f_n(t)|\,dt.$

<u>IF</u> $\displaystyle\sum_{n=0}^{\infty}\int_a^b|f_n(t)|\,dt < \infty$ THEN

$$\sum_{n=0}^{\infty}\int_a^b f_n(t)\,dt = \int_a^b\sum_{n=0}^{\infty}f_n(t)\,dt$$

Cor $\{f_n\}_{n\geq 0}$ seq. of cont. fcns $f_n : D \longrightarrow \mathbb{C}$, $\gamma \subseteq D$ oriented curve. Let $M_n = \max_{z\in\gamma} |f_n(z)|$.

IF $\sum_{n=0}^{\infty} M_n < \infty$ and $\text{length}(\gamma) < \infty$ then

$$\sum_{n=0}^{\infty} \int_\gamma f_n(z)\,dz = \int_\gamma \sum_{n=0}^{\infty} f_n(z)\,dz$$

Proof

$$\sum_{n=0}^{\infty} \int_\gamma f_n(z)\,dz = \sum_{n=0}^{\infty} \int_a^b f_n(\gamma(t))\,\gamma'(t)\,dt$$

WANT!

$$\sum_{n=0}^{\infty} \int_a^b \left|f_n(\gamma(t))\,\gamma'(t)\right| dt \leq \sum_{n=0}^{\infty} \left(\text{length}(\gamma)\cdot M_n\right) < \infty$$

$\square$

Thm $f : D \longrightarrow \mathbb{C}$ analytic, $z_0 \in D$, $B(z_0,R) \subseteq D$.

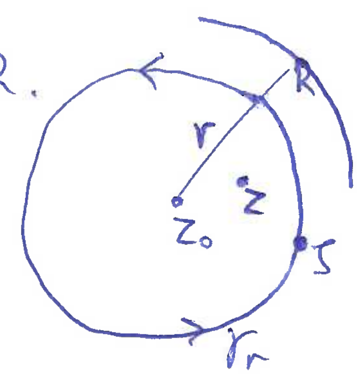
Then $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for } z \in B(z_0,R)$$

where $$a_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \qquad 0<r<R$$

Proof

Note: Green $\Rightarrow$ a_n is indep. of r.

Let $z \in B(z_0,R)$. Choose r s.t. $|z| < r < R$.



$$f(z) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)-(z-z_0)} = \frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\cdot\frac{1}{\zeta-z_0}$$

$$= \frac{1}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \qquad \text{since } \left|\frac{z-z_0}{\zeta-z_0}\right| < 1.$$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{n=0}^{\infty} \frac{f(\zeta)\,(z-z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \right) (z-z_0)^n$$

<u>Note:</u> $M_n = \frac{1}{|z-z_0|} \left| \frac{z-z_0}{\zeta - z_0} \right|^{n+1} \left(\max_{\zeta \in \gamma_r} |f(\zeta)| \right)$

So $\sum_{n=0}^{\infty} M_n < \infty$.

$\square$

Complex Analysis 2/27/2018 ①

2.4 Consequences of Cauchy's formula (cont.)

Last time:

$f: D \to \mathbb{C}$ analytic, $z_0 \in D$, $B(z_0, R) \subseteq D$.

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $z \in B(z_0, R)$

where $a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}} dz$, $0 < r < R$

$\boxed{\text{Recall: } f^{(k)}(z_0) = k!\, a_k}$

Cor $f: D \to \mathbb{C}$ analytic $\Rightarrow$ $f^{(k)}(z)$ exists and is analytic on D $\forall\, k \geq 0$.

Cor $f: D \to \mathbb{C}$ analytic, $D \subseteq \mathbb{C}$ connected open subset.

let $z_0 \in D$. Assume $f^{(k)}(z_0) = 0$ for all $k \geq 0$.

Then $f(z) = 0$ $\forall\, z \in D$.

Idea:

Choose $R > 0$ s.t. $B(z_0, R) \subseteq D$.

Then $f(z) = 0$ for all $z \in B(z_0, R)$ by p.s. exp.

$\exists$ largest open subset Ω of D s.t. $z_0 \in \Omega$ and $f(z) = 0$ $\forall z \in \Omega$.

~~Bcp~~ If $\Omega \subsetneq D$ then let $z \in D \setminus \Omega$.

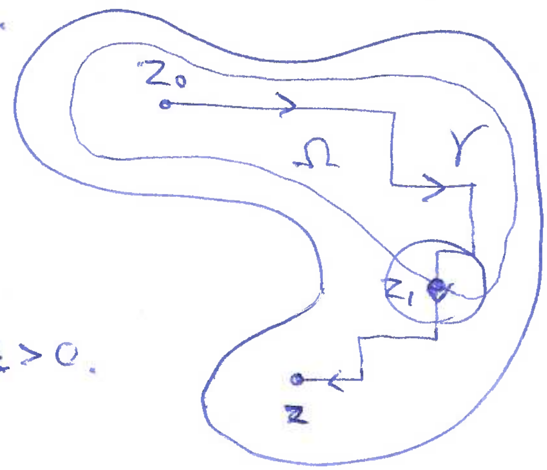
let γ be curve from z_0 to z.

let $z_1 \in \gamma$ be "first point outside Ω".

Must have $f^{(k)}(z_1) = 0$ $\forall\, k \geq 0$,

So $f(z) = 0$ on $B(z_1, \varepsilon)$ for some $\varepsilon > 0$.

Now $\Omega \subsetneq \Omega \cup B(z_1, \varepsilon)$ ↯.



□

$\boxed{\text{Q: Is Cor 1 true for real functions?}}$

Cor 2

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

∞-diff on $\mathbb{R}$.

$f^{(k)}(0) = 0$ $\forall k \geq 0$

Order of zero

Let $f: D \to \mathbb{C}$ be analytic, NOT identically zero.

Let $z_0 \in D$ and write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for } z \in B(z_0, R) \subseteq D.$$

Choose m such that $a_0 = a_1 = \dots = a_{m-1} = 0$, $a_m \neq 0$

minimal s.t. $a_m \neq 0$, i.e.

Def We say $f(z)$ has zero of order m at z_0.

Equivalent: $f^{(k)}(z_0) = 0$ for $0 \le k \le m-1$
and $f^{(m)}(z_0) \neq 0$.

Examples zero of order 0: $f(z_0) \neq 0$ — Not zero! 1, $\exp(z)$

zero of order 1: $f(z_0) = 0$, $f'(z_0) \neq 0$. $z - z_0 = f(z)$

zero of order 2: $f(z_0) = f'(z_0) = 0$, $f''(z_0) \neq 0$. $f(z) = (z-z_0)^2$

Thm If $f(z)$ has zero of order m at $z_0 \in D$, then we can write $f(z) = (z-z_0)^m g(z)$, where $g: D \to \mathbb{C}$ is analytic and $g(z_0) \neq 0$.

Proof

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n, \quad a_m \neq 0.$$

$$g(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^{n-m}, \quad g(z_0) = a_m \neq 0.$$

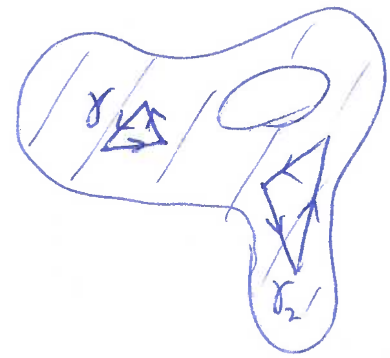
$\square$

Morera's Thm

$D \subseteq \mathbb{C}$ open, $f: D \longrightarrow \mathbb{C}$ continuous.

Assume $\int_\gamma f(z)\,dz = 0$ for every triangle $\gamma \subseteq D$ such that points inside $\gamma \subseteq D$.

Then f is analytic on D.



Pf

Let $z_0 \in D$.

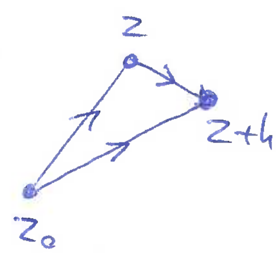
Choose $R > 0$ s.t. $B(z_0, R) \subseteq D$.

WLOG $D = B(z_0, R)$.

Def. $F: B(z_0, R) \longrightarrow \mathbb{C}$ by

$$F(z) = \int_{z_0}^{z} f(z)\,dz$$ — line integral along line segment $z_0 \longrightarrow z$

Idea: $F(z)$ analytic with $F'(z) = f(z)$.



Note: $$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\,d\zeta$$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_z^{z+h} \big(f(\zeta) - f(z)\big)\,d\zeta \right|$$

$$\leq \frac{1}{|h|} \cdot \underbrace{\mathrm{dist}(z, z+h)}_{|h|} \cdot \max_{Z} |f(\zeta) - f(z)| \longrightarrow 0 \text{ as } h \to 0$$

Z — on line segment $z \to z+h$

$\square$

Liouville's Thm

$f : \mathbb{C} \to \mathbb{C}$ analytic (entire).

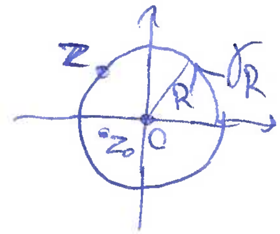
Assume $\exists\, M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

Then $f(z)$ is constant fcn.

Proof

$g(z) = \dfrac{F(z) - F(0)}{z}$ is entire, and $|g(z)| \leq \dfrac{2M}{|z|}$ for $z \neq 0$.

Let $z_0 \in \mathbb{C}$. For $R > |z_0|$ we have



$$g(z_0) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{g(z)}{z - z_0}\, dz$$

$$|g(z_0)| \leq \frac{1}{2\pi}\, \text{length}(\gamma_R)\, \frac{2M/R}{R - |z_0|} = \frac{2M}{R - |z_0|} \longrightarrow 0 \text{ as } R \to \infty.$$

$\therefore\ g(z_0) = 0 \quad \forall\, z_0 \in \mathbb{C}.$

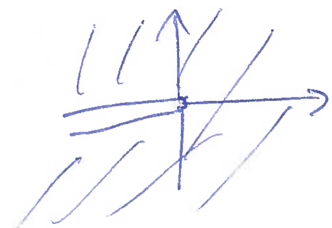
$f(z)$ constant.

$\square$

Analytic Logarithm

Thm $f : D \longrightarrow \mathbb{C}$ analytic.
Assume D simply connected and $f(z) \neq 0 \;\; \forall z \in D$.
Then $\exists$ analytic $g : D \longrightarrow \mathbb{C}$ such that
$f(z) = \exp(g(z)) \quad \forall z \in D$.

Example $f(z) = z$, $D = \mathbb{C} - \mathbb{R}_{\leq 0}$.



$$z = \exp(g(z))$$

$$g(z) = \log(z).$$

Proof The function $\dfrac{f'(z)}{f(z)}$ is analytic on D.

Simply conn $\Rightarrow \exists$ antiderivative $h : D \rightarrow \mathbb{C}$, $h' = \dfrac{f'}{f}$.

$\left[h(z) = \displaystyle\int_{z_0}^{z} \frac{f'(z)}{f(z)} dz \right]$ for some choice of $z_0 \in D$.

$$\left(e^{-h(z)} f(z) \right)' = -h'(z)\, e^{-h(z)} f(z) + e^{-h(z)} f'(z)$$

$$= -f'(z)\, e^{-h(z)} + e^{-h(z)} f'(z) = 0.$$

$\therefore \;\; f(z)\, e^{-h(z)} = c$ constant.

$$f(z) = c \exp(h(z)) = \exp(g(z)),$$

$$g(z) = h(z) + \operatorname{Log}(c).$$

$\square$

Multiply Power series

$$\left.\begin{aligned} f(z) &= \sum a_n z^n \\ g(z) &= \sum b_n z^n \end{aligned}\right\} \quad \text{on } B(0,R)$$

$h(z) = f(z)\cdot g(z)$ analytic. on $B(0,R)$.

$$h(z) = \sum c_n z^n.$$

<u>Claim:</u> $\quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$

$$c_n = \frac{1}{n!} h^{(n)}(0).$$

$$h^{(n)}(0) = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} f^{(k)}(0)\, g^{(n-k)}(0)$$

$$c_n = \sum \frac{f^{(k)}(0)}{k!} \frac{g^{(n-k)}(0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}.$$

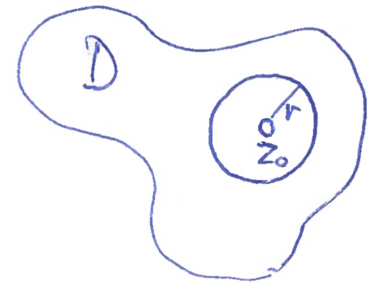
2.5 Isolated singularities

$\boxed{\text{Comment on Midterm 1}}$

$f: D \longrightarrow \mathbb{C}$ analytic.

Assume $z_0 \notin D$ but $B(z_0, r) - \{z_0\} \subseteq D$ for some $r > 0$

We say f has isolated singularity at z_0.



Examples

$f(z) = \frac{1}{z}$ has isolated sing at $z_0 = 0$

$f(z) = \frac{1}{\sin(z)}$ has isolated sing at $k\pi$ for each $k \in \mathbb{Z}$

$f(z) = \frac{z^2-1}{z-1}$ has isolated sing at $z_0 = 1$. But $f(z) = z+1$.

Set $D = \mathbb{C} - \{0\}$, def. $f: D \longrightarrow \mathbb{C}$ by $f(z) = 0$.

Then $f(z)$ has isolated sing at 0. (!)

Def f has removable singularity at z_0 if z_0 isolated sing and $|f(z)|$ is bounded as $z \to z_0$.

$\exists M > 0$ and $\exists r > 0$ s.t. $|f(z)| < M$ for all $z \in B(z_0, r) - \{z_0\}$

(and $B(z_0, r) - z_0 \subseteq D$.)

Remove removable singularity Assume z_0 removable sing of $f(z)$:

$$g(z) = \begin{cases} (z-z_0)^2 f(z). & \text{if } 0 < |z-z_0| < r \\ 0 & \text{if } z = z_0 \end{cases}$$

g is analytic on $B(z_0, r) - z_0$

$$\left|\frac{g(z) - g(z_0)}{z - z_0}\right| = \left|\frac{(z-z_0)^2 f(z) - 0}{z - z_0}\right| = |(z-z_0) f(z)| \le |z-z_0| \cdot M \xrightarrow[z \to z_0]{} 0$$

$\therefore$ g analytic on $B(z_0, r)$, $g(z_0) = g'(z_0) = 0$.

g has zero of order ≥ 2 at z_0.

$g(z) = (z-z_0)^2 h(z)$, $h : B(z_0, r) \to \mathbb{C}$ analytic.

$h(z) = f(z)$ for $z \neq z_0$.

$\therefore$ f can be extended to analytic fcn $D \cup \{z_0\} \to \mathbb{C}$
by setting $f(z_0) = h(z_0)$.

Example $f(z) = \dfrac{\sin(z)}{z}$ removable sing at $z_0 = 0$.

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \qquad \text{valid at } z = 0.$$

Poles

Def f has a pole at z_0 if z_0 isolated sing and
$|f(z)| \to +\infty$ as $z \to z_0$

Example $f(z) = \dfrac{\exp(z)}{(z-z_0)^m}$

Assume $f(z)$ has pole at z_0.
Choose $r > 0$ so that f is analytic on $B(z_0, r) \setminus \{z_0\}$
and $|f(z)| \geq 1$ for $z \in B(z_0, r) \setminus z_0$

$g(z) = \dfrac{1}{f(z)}$ for $z \in B(z_0, r) - \{z_0\}$

g analytic on $\overset{*}{B}(z_0, r) := B(z_0, r) - \{z_0\}$. (since $f(z) \neq 0$.)

$|g(z)| \leq 1$ bounded on $B^*(z_0, r)$.

$\therefore$ g has removable sing at z_0.

Exten g to analytic fcn. $g : B(z_0, r) \to \mathbb{C}$.

Note: $g(z_0) = 0$ since $|g(z)| = \frac{1}{|f(z)|} \to 0$ as $z \to z_0$. ③

$g(z)$ has zero of order m at z_0, some $m \geq 1$.

$g(z) = (z-z_0)^m h(z)$, $h: B(z_0, r) \to \mathbb{C}$ analytic,
$h(z_0) \neq 0$.

Note: $f(z) \neq 0$ on $B^*(z_0, r) \Rightarrow g(z) \neq 0$ on $B^*(z_0, r)$
$\Rightarrow h(z) \neq 0$ on $B(z_0, r)$.

$H(z) = \frac{1}{h(z)}$ analytic on $B(z_0, r)$.

$$\boxed{f(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^m h(z)} = \frac{H(z)}{(z-z_0)^m}, \quad H \text{ analytic. } H(z_0) \neq 0.}$$

Say: f has pole of order m at z_0.

Essential Singularity:

Isolated sing that is not removable and not pole.
$|f(z)|$ not bounded as $z \to z_0$, but $|f(z)| \not\to +\infty$ as $z \to z_0$.

Example $f(z) = \exp\left(\frac{1}{z}\right)$ has essential sing. at $z_0 = 0$.

Note: For $x \in \mathbb{R}$: $\exp\left(\frac{1}{x}\right) \to 0$ as $x \to 0^-$
$\exp\left(\frac{1}{x}\right) \to +\infty$ as $x \to 0^+$

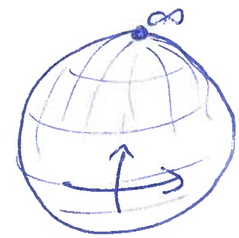
Example Assume f has essential sing at z_0 and let $w \in \mathbb{C}$.
Then $\exists$ sequence $\{z_n\}$ such that
$z_n \to z_0$ and $f(z_n) \to w$ as $n \to \infty$.

Else $|f(z) - w| \geq M > 0$ bounded from below as $z \to z_0$.

$\Rightarrow$ $g(z) = \frac{1}{f(z) - w}$ has removable sing. at z_0 $\Rightarrow$ $f(z) = \frac{1}{g(z)} + w$
has ~~rem.~~ sing / pole @ z_0.

Remark Assume $f: D \longrightarrow \mathbb{C}$ has isolated sing at z_0.

$$f: D \longrightarrow \mathbb{C} \subseteq \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$



essential singularity:

$f(z)$ does not converge to point in $\mathbb{CP}^1$ as $z \to z_0$

removable sing:

$f(z) \longrightarrow w$ as $z \to z_0$ for some $w \in \mathbb{C}$

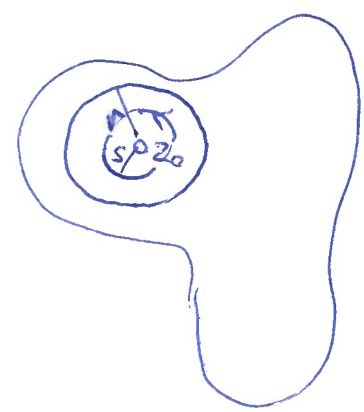
pole:

$f(z) \longrightarrow \infty$ in $\mathbb{CP}^1$ as $z \to z_0$.

Residue at z_0

$f: D \longrightarrow \mathbb{C}$ analytic with isolated sing @ z_0.

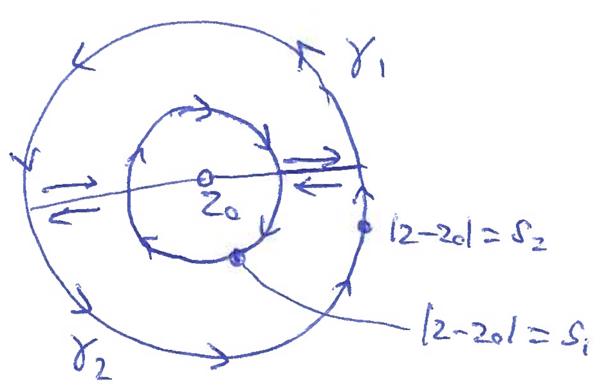
$B^*(z_0, r) \subseteq D$.



Def
$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} f(z)\, dz$$

where $0 < s < r$.

Well defined: $0 < s_1 < s_2 < r$



Cauchy's Thm $\Rightarrow$

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz = 0$$

Residue at removable singularity z_0:

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} f(z)\, dz = 0 \quad \text{by Cauchy's Thm.}$$

Residue at pole z_0

Assume $f(z)$ has pole of order $m > 0$ at z_0.

$$f(z) = \frac{H(z)}{(z-z_0)^m}, \quad H \text{ analytic on } B(z_0, r), \; H(z_0) \neq 0.$$

$$H(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k \quad \text{valid on } B(z_0, r).$$

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} \left(\sum_{k=0}^{\infty} c_k (z-z_0)^{k-m} \right) dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{|z-z_0|=s} c_k (z-z_0)^{k-m}\, dz$$

$$= c_{m-1}$$

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{H(z)}{(z-z_0)^m}\, dz = c_{m-1}$$

$$= \frac{H^{(m-1)}(z_0)}{(m-1)!}$$

Example $f(z) = \dfrac{\sin(z)}{z^4}$

$$\sin(z) = \sum (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\operatorname{Res}(f;0) = \left[\text{coef of } z^3 \text{ in } \sin(z)\right] = -\frac{1}{6}$$

Example $f(z) = \dfrac{\sin(z)}{(z-1)^4}$

$$\operatorname{Res}(f;1) = \frac{\sin'''(1)}{3!} = \frac{-\cos(1)}{6}$$