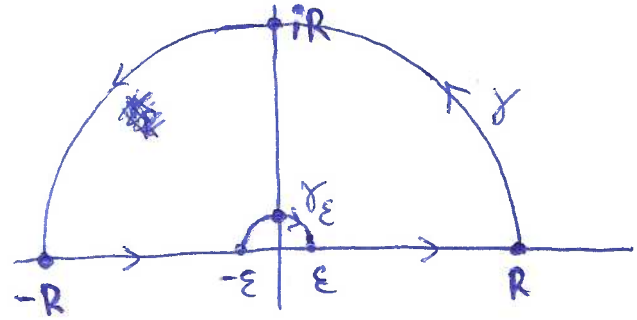
2.6 Example 6 $\displaystyle\int_0^\infty \frac{\sin(x)^2}{x^2}\,dx$

$\boxed{\text{Review + Catch-up}}$ $\boxed{\text{MT2 4/3}}$

$$(\sin x)^2 = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2}\operatorname{Re}\left(1 - e^{2ix}\right)$$

$$f(z) = \frac{1 - e^{2iz}}{z^2}$$

$$\int_\gamma f(z)\,dz = 0$$



Note: If $|z| = R$, $\operatorname{Im}(z) \geq 0$

then $\operatorname{Re}(2iz) \leq 0$, so $\displaystyle |f(z)| \leq \frac{1 + |e^{2iz}|}{|z^2|} \leq \frac{2}{R^2}$

$$\left|\int_{\substack{|z|=R\\ \operatorname{Im}(z)\geq 0}} f(z)\,dz\right| \leq R\pi \cdot \frac{2}{R^2} = \frac{2\pi}{R} \longrightarrow 0 \quad \text{as } R \to \infty.$$

$$f(z) = \frac{1 - \sum_{n=0}^\infty \frac{(2iz)^n}{n!}}{z^2} = -\sum_{n=1}^\infty \frac{(2iz)^n}{z^2 n!} = 4\sum_{m=-1}^\infty \frac{(2iz)^m}{(m+2)!}$$

$$\gamma_\varepsilon(t) = \varepsilon e^{it}, \quad \pi \geq t \geq 0.$$

$$\gamma_\varepsilon'(t) = i\,\gamma_\varepsilon(t)$$

$$\int_{\gamma_\varepsilon} z^m\,dz = \int_\pi^0 \gamma_\varepsilon(t)^m\, i\,\gamma_\varepsilon(t)\,dt = i\,\varepsilon^{m+1}\left[\frac{e^{it(m+1)}}{i(m+1)}\right]_\pi^0$$

$$\longrightarrow 0 \quad \text{as } \varepsilon \to 0, \qquad \boxed{m \geq 0}$$

$$m = -1: \quad 4\frac{(2iz)^m}{(m+2)!} = \frac{-2i}{z}$$

$$\int_{\gamma_\varepsilon}\left(-\frac{2i}{z}\right)dz = -\int_\pi^0 \frac{2i}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt = \int_\pi^0 2\,dt = -2\pi.$$

$$\therefore \int_{\gamma_\varepsilon} f(z)\,dz \longrightarrow -2\pi \quad \text{as } \varepsilon \to 0$$

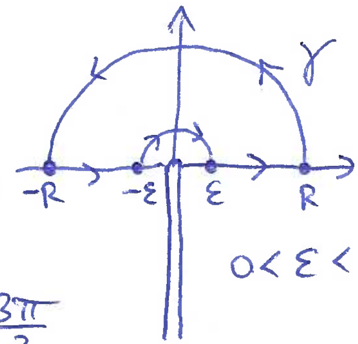
$$\therefore \int_\varepsilon^R \frac{\sin(x)^2}{x^2}\,dx \to \frac{\pi}{2} \quad \text{as } \varepsilon \to 0,\ R \to \infty.$$

Example 8 $\displaystyle\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx$

$$f(z) = \frac{\log(z)}{(1+z^2)^2}$$

use branch of log on $\mathbb{C}$ − negative imaginary axis.

$$-\frac{\pi}{2} < \operatorname{Im}(\log(z)) < \frac{3\pi}{2}.$$



$0 < \varepsilon < 1 < R$

Singularities inside γ: $z = i$

$$f(z) = \frac{h(z)}{(z-i)^2}, \qquad h(z) = \frac{\log(z)}{(z+i)^2}$$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{h(z)}{(z-i)^2}\,dz = h'(i)$$

$$h'(z) = \frac{\frac{1}{z}(z+i)^2 - \log(z)\cdot 2(z+i)}{(z+i)^4}$$

$$h'(i) = \frac{-i(2i)^2 - (\frac{\pi}{2}i)\cdot 2\cdot 2i}{(2i)^4} = \frac{4i+2\pi}{16}$$

$$\int_\gamma f(z)\,dz = 2\pi i\,\frac{4i+2\pi}{16} = -\frac{\pi}{2} + \frac{i\pi^2}{4}$$

Estimates:

$|z| = R$: $\quad |f(z)| = \dfrac{|\log(z)|}{|z^2+1|^2} \leq \dfrac{2\log R}{(R^2-1)^2}$, R large.

$$\left|\int_{\substack{|z|=1\\ \operatorname{Im}(z)\geq 0}} f(z)\,dz\right| \leq \pi R\cdot\frac{2\log(R)}{(R^2-1)^2} \longrightarrow 0 \text{ as } R\to\infty.$$

$|z| = \varepsilon$: $\quad |f(z)| \leq \dfrac{-2\log(\varepsilon)}{(1-\varepsilon^2)^2}$

$$\left|\int_{\substack{|z|=\varepsilon\\ \operatorname{Im}(z)\geq 0}} f(z)\,dz\right| \leq \pi\varepsilon\,\frac{-2\log(\varepsilon)}{(1-\varepsilon^2)^2} \longrightarrow 0 \text{ as } \varepsilon\to 0$$

Note: $-R < x < -\varepsilon$: $\quad f(x) = \dfrac{\log|x| + i\pi}{(x^2+1)^2}$

$$\therefore \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\int_{-R}^{-\varepsilon} \frac{\log(-x)+i\pi}{(x^2+1)^2}\,dx + \int_{\varepsilon}^{R} \frac{\log(x)}{(x^2+1)^2}\,dx \right] = \int_\gamma f(z)\,dz = -\frac{\pi}{2} + i\frac{\pi^2}{4}$$

Real part: $\displaystyle\int_0^\infty \frac{\log(x)}{(x^2+1)^2}\,dx = -\frac{\pi}{4}$

Imaginary part: $\displaystyle\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{4}$

Example $\quad f(z) = \dfrac{z^2+7}{(z-1)^2(z+3)}$

Find Laurent series valid on annulus $\quad 1 < |z| < 3$

$$f(z) = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+3}$$

Solve : $\quad A = 0, \quad B = 2, \quad C = 1.$

$$f(z) = \frac{2}{(1-z)^2} + \frac{1}{3+z}$$

$$\frac{1}{3+z} = \frac{1}{3}\,\frac{1}{1+\frac{z}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}\,z^n$$

since $\left|\frac{z}{3}\right| < 1.$

$$\frac{2}{(z-1)^2} = \frac{2}{z^2}\cdot\frac{1}{(1-z^{-1})^2} = \frac{2}{z^2}\left(\sum_{n=0}^{\infty} z^{-n}\right)^2$$

$$= \frac{2}{z^2}\sum_{n=0}^{\infty}(n+1)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} 2(n+1)\,z^{-n-2}$$

$m = -n-2$

$n = -m-2$

$$= \sum_{m=-\infty}^{-2} 2(-m-1)\,z^m$$

$$f(z) = \sum_{m=-\infty}^{-2} 2(-n-1)\,z^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}\,z^n$$

3.3 Linear Fractional Transformations

Linear Fractional Transform:

$$T(z) = \frac{az+b}{cz+d}$$ where $a,b,c,d \in \mathbb{C}$ constants, $ad-bc \neq 0$.

Note: $$T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

so $T(z)$ Not constant.

Note: Pole at $z = -\frac{d}{c}$, zero at $z = -\frac{b}{a}$

$T(z) \longrightarrow \frac{a}{c}$ as $z \to \infty$.

Def $T : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$, $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

$$T(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C},\ cz+d \neq 0 \\ \infty & \text{if } z \in \mathbb{C},\ cz+d = 0 \\ a/c & \text{if } z = \infty,\ c \neq 0 \\ \infty & \text{if } z = \infty,\ c = 0 \end{cases}$$

Claim: T is bijective.

$$u(z) = \frac{-dz+b}{cz-a}$$

$$u(T(z)) = \frac{-d\,\frac{az+b}{cz+d} + b}{c\,\frac{az+b}{cz+d} - a} = z$$

$$T(u(z)) = z$$

u inverse transform.

Example $T(z) = \lambda \dfrac{z-a}{\bar{a}z-1} = \dfrac{\lambda z - \lambda a}{\bar{a}z-1}$, $a,\lambda \in \mathbb{C}$, $|a|<1$, $|\lambda| = 1$.

Claim: T maps unit ball $\Delta = B(0,1)$ onto itself.

If $|z|<1$ then

$$|T(z)|^2 = |\lambda|^2 \frac{(z-a)(\bar{z}-\bar{a})}{(\bar{a}z-1)(a\bar{z}-1)} = \frac{|z|^2+|a|^2-2\operatorname{Re}(\bar{a}z)}{|a|^2|z|^2+1-2\operatorname{Re}(\bar{a}z)} < 1$$

because $|z|^2+|a|^2 < |a|^2|z|^2+1$

because $|a|^2|z|^2-|a|^2-|z|^2+1 = (1-|a|^2)(1-|z|^2) > 0$.

$\therefore T(\Delta) \subseteq \Delta$

$$T^{-1}(z) = \frac{z-\lambda a}{\bar{a}z-\lambda} = \bar{\lambda}\,\frac{z-\lambda a}{\overline{\lambda a}\,z-1}$$ same type of fcn.

$\therefore T^{-1}(\Delta) \subseteq \Delta$.

So if $w \in \Delta$ then set $z = T^{-1}(w) \in \Delta$. $T(z) = T(T^{-1}(w)) = w$.

Fixed points.

$T(z) = \dfrac{az+b}{cz+d}$ Lin. Frac. Trans. $\boxed{\text{Assume } T(z) \neq z.}$

Claim: At most two points $z \in \mathbb{CP}^1$ for which $T(z) = z$.

Assume $T(\infty) = \infty$.

Then $c = 0$. $T(z) = \frac{a}{d}z + \frac{b}{d}$.

$T(z) = z$ has at most one solution $z \in \mathbb{C}$.

Assume $T(\infty) \neq \infty$. Then $c \neq 0$.

$$T(z) = z \iff az+b = z(cz+d)$$

$$\iff cz^2 + (d-a)z - b = 0$$

has at most 2 sols. $z \in \mathbb{C}$.

Thm If $T(z) = \dfrac{az+b}{cz+d}$ has 3 distinct fixed points in $\mathbb{CP}^1$ then $T(z) = z$ for all $z \in \mathbb{CP}^1$.

Example

$T(z) = 2z$ has 2 fixed points: 0 and ∞.

$T(z) = z+1$ has 1 fixed point: ∞

$T(z) = \dfrac{z+2}{2z+1}$ has 2 fixed points: ± 1.

$T(z) = \dfrac{z}{z+1}$ has 1 fixed point: 0

Determined by 3 values

Assume $T(z)$ and $S(z)$ are Frac. Lin. Trans such that $T(z_j) = S(z_j)$ for 3 distinct points $z_1, z_2, z_3 \in \mathbb{CP}^1$. Then $S(z) = T(z)$ $\forall z \in \mathbb{CP}^1$.

$S^{-1}\circ T$ is a Frac. Lin. Trans with 3 fixed points z_1, z_2, z_3

So $S^{-1}\circ T = id \Rightarrow S = T$.

Q: Given $z_1, z_2, z_3 \in \mathbb{CP}^1$ distinct, and $w_1, w_2, w_3 \in \mathbb{CP}^1$ distinct.

Can we find $T: \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $T(z_j) = w_j$?

Construct $U: \mathbb{CP}^1 \to \mathbb{CP}^1$ such that

$U(z_1) = 0, \quad U(z_2) = 1, \quad U(z_3) = \infty.$

$$U(z) = \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right)$$

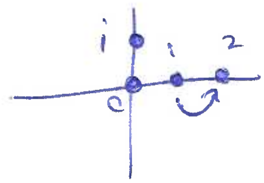
LEAVE OUT ANY FACTORS INVOLVING ∞!

Similarly, Construct S with $S(w_1) = 0 \quad S(w_2) = 1 \quad S(w_3) = \infty$.

$T = S^{-1}U$ satisfies $T(z_j) = w_j$.

Example Find $T: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ such that

$$T(0) = 0, \quad T(i) = i, \quad T(1) = 2$$



$S:$ $S(0) = 0, \quad S(i) = \infty, \quad S(1) = 1$

$$S(z) = \frac{z-0}{z-i}\,\frac{1-i}{1-0} = (1-i)\frac{z}{z-i}$$

$U:$ $U(0) = 0, \quad U(i) = \infty, \quad U(2) = 1$

$$U(z) = \frac{z-0}{z-i}\,\frac{2-i}{2-0} = \frac{2-i}{2}\,\frac{z}{z-i}$$

$$T = U^{-1}S$$

$$U^{-1}(z) = \frac{2iz}{+2z-2+i}$$

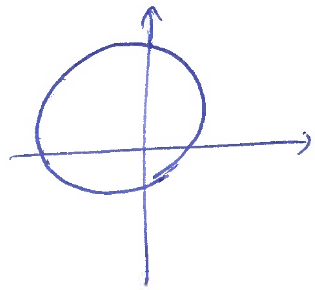
$$T(z) = U^{-1}(S(z)) = \frac{(-2+2i)\,z}{z+(-2+i)}$$

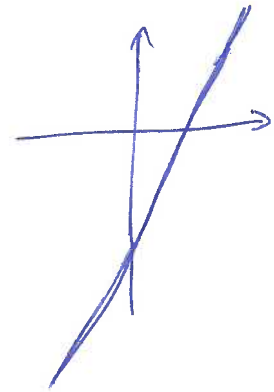
$$T(0) = 0, \quad T(i) = i, \quad T(1) = 2.$$

Circles and lines

$$T(z) = \frac{az+b}{cz+d}.$$

Let $F \subseteq \mathbb{CP}^1$ be a circle or line.





Claim: $T(F)$ is a circle or a line.

Note $T(z) = \frac{1}{c}\left(\frac{bc-ad}{cz+d} + a\right)$

$T(z) = T_6 T_5 T_4 T_3 T_2 T_1 (z)$

$T_1(z) = cz$

$T_2(z) = z + d$

$T_3(z) = \frac{1}{z}$

$T_4(z) = (bc-ad)z$

$T_5(z) = z + a$

$T_6(z) = \frac{1}{c} z$

Case 1

$T(z) = cz$, $c \in \mathbb{C}$, $c \neq 0$.

scale by $|c|$

+ rotate by $\arg(c)$.

$T(F)$ circle / line.

Case 2

$T(z) = z + d$.

Translate by d.

$T(F)$ circle / line.

Case 3

$T(z) = \frac{1}{z}$.

$F: \alpha(x^2+y^2) + \beta x + \gamma y + \delta = 0$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$. $(\alpha, \beta, \gamma) \neq (0,0,0)$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u + iv, \quad u = \frac{x}{x^2+y^2} \quad v = \frac{-y}{x^2+y^2}$$

$$\boxed{u^2+v^2 = \frac{1}{x^2+y^2}}$$

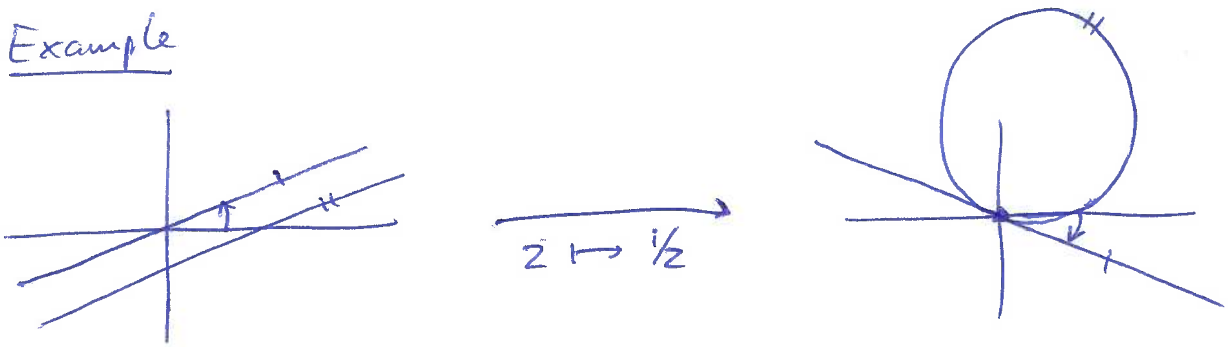
$$\alpha(x^2+y^2) + \beta x + \gamma y + \delta = 0$$

$$\alpha + \beta \frac{x}{x^2+y^2} + \gamma \frac{y}{x^2+y^2} + \delta \frac{1}{x^2+y^2} = 0$$

$$\alpha + \beta u - \gamma v + \delta (u^2+v^2) = 0$$

$u+iv = T(z) = \frac{1}{z}$ is on line / circle.

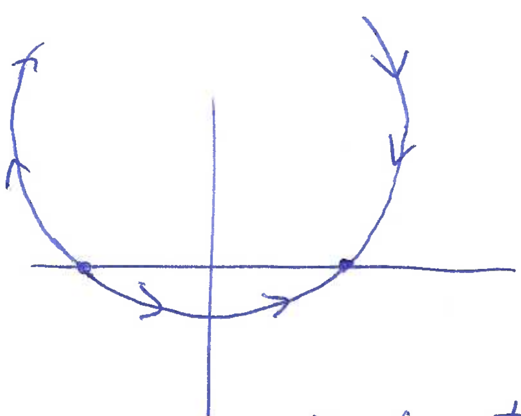
Example



Example $\quad T(z) = \dfrac{z+2}{2z+1}.$

Fixed points : $\quad T(1) = 1, \quad T(-1) = -1.$

Can show : T ~~fixes~~ maps any circle through ± 1 to itself.



And: $T^n(z) = T(T(T(\cdots(T(z))\cdots)))$

$T^n(z_0) \longrightarrow 1$ as $n \to \infty$

if $z_0 \neq -1$.

Explanation : $\quad U(z) = \dfrac{z+1}{z-1}$

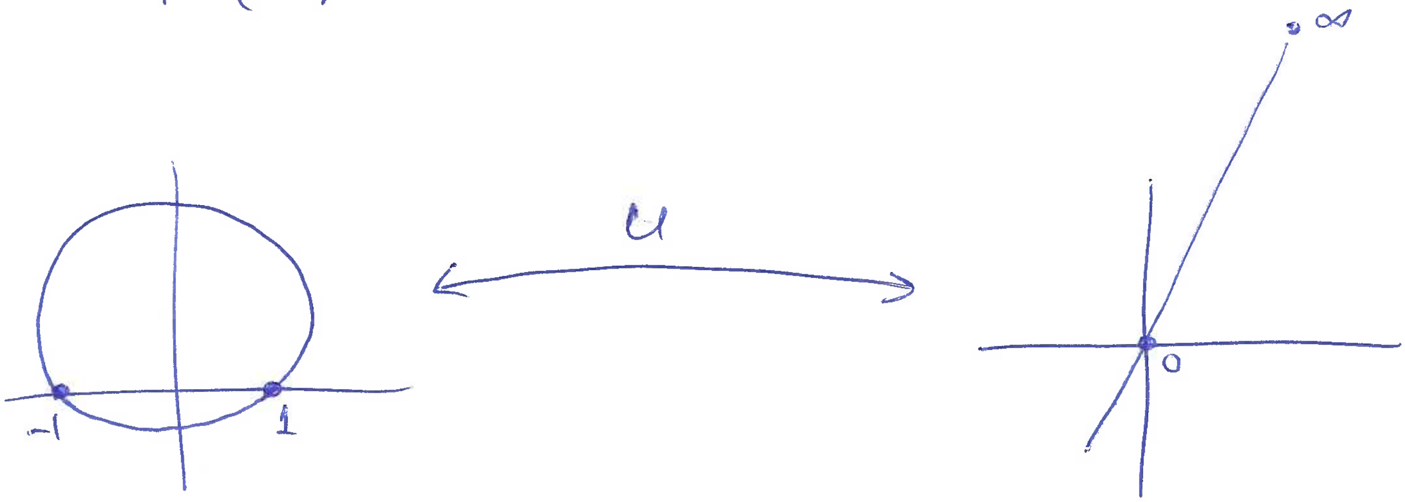
$U(-1) = 0, \quad U(1) = \infty. \qquad U^{-1}(z) = U(z).$

Consider $\tilde{T}(z) = u(T(u(z)))$

Check: $\tilde{T}(z) = -3z$.

$\tilde{T}$ maps any line through 0 and ∞ to itself.

$\tilde{T}^n(z_0) \longrightarrow \infty$ as $n \to \infty$ if $z_0 \neq 0$.



WARNING: Transformations with 2 fixed pts are more complicated than indicated in book.

Assume

$T: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ has fixed points $0, \infty$.

Then $T(z) = \alpha z$, $\alpha \in \mathbb{C}$, $\alpha \neq 0$.

preserves lines through $0, \infty$ $\iff$ $\alpha \in \mathbb{R}$.

$T^n(z_0) \to \infty$ if $|\alpha| > 1$

$T^n(z_0) \longrightarrow 0$ if $|\alpha| < 0$

Neither if $|\alpha| = 1$.

Exams back.

Laurent series of $f(z) = \frac{1}{\tan(z)} = \frac{\cos(z)}{\sin(z)}$ on $\overset{*}{B}(0, \pi)$.

$\cos(0) = 1 \neq 0$, $\sin(z)$ has zero of mult. one at 0.

$\Rightarrow \frac{\cos(z)}{\sin(z)}$ has simple pole (order 1) at 0.

$$f(z) = a_{-1} z^{-1} + a_0 + a_1 z + \cdots = \sum_{-1}^{\infty} a_n z^n$$

$$f(z) \sin(z) = \cos(z)$$

$$\left(a_{-1} z^{-1} + a_0 + a_1 z + \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)$$

LHS: $(a_{-1}) + (a_0) z + \left(a_1 - \frac{a_{-1}}{3!}\right) z^2 + \cdots$

$$a_{-1} = 1$$

$$a_0 = 0$$

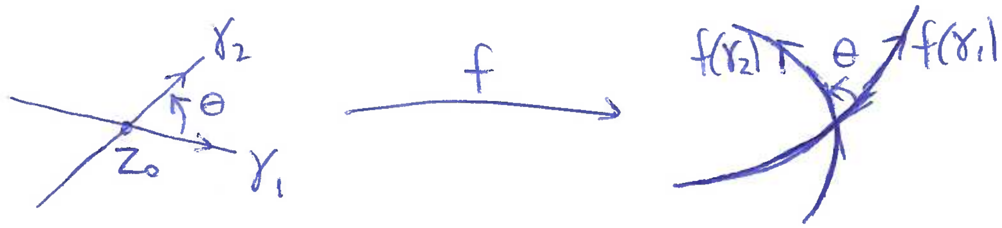
$$a_1 - \frac{a_{-1}}{3!} = -\frac{1}{2!} \quad \Rightarrow \quad a_1 = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$$

$$\frac{\cos(z)}{\sin(z)} = \frac{1}{z} - \frac{1}{3} z + \cdots$$

principal part at $z_0 = 0$: $P\left(\frac{1}{z}\right) = \frac{1}{z}$.

3.4 Conformal maps

A function $f: D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, is **conformal** at $z_0 \in D$ if it preserves the angle between two curves that meet at z_0.



$\gamma_1 : (-\varepsilon, \varepsilon) \to D$, $\quad \gamma_1(0) = z_0$

$\gamma_2 : (-\varepsilon, \varepsilon) \to D$, $\quad \gamma_2(0) = z_0$

Tangent directions: $\gamma_1'(0)$ and $\gamma_2'(0)$

$$\theta = \arg\left(\frac{\gamma_2'(0)}{\gamma_1'(0)}\right)$$

$\therefore$ f is conformal at z_0 if $\arg\left(\dfrac{\gamma_2'(0)}{\gamma_1'(0)}\right) = \arg\left(\dfrac{(f\gamma_2)'(0)}{(f\gamma_1)'(0)}\right)$

Assume f is analytic at z_0, $f'(z_0) \neq 0$.

$$(f\gamma_i)'(0) = f'(z_0)\gamma_i'(0). \quad \Rightarrow \quad \frac{\gamma_2'(0)}{\gamma_1'(0)} = \frac{(f\gamma_2)'(0)}{(f\gamma_1)'(0)}.$$

Thm If f is analytic at z_0, with $f'(z_0) \neq 0$, then f is conformal at z_0

Recall: If $f'(z_0) = 0$ then $f(z) - f(z_0)$ has zero of order $m \geq 2$.

$\Rightarrow$ $f(z)$ is m-to-one for z close to z_0.

Example $f(z) = z^3$ is 3-to-1 close to $z_0 = 0$

Thm If f is both analytic and one-to-one on $D \subseteq \mathbb{C}$, then $f(z)$ is conformal at all points of D.

Examples 1) $\exp(z)$ is conformal at all points $z_0 \in \mathbb{C}$.

2) Any frac. Lin. trans $f(z) = \dfrac{az+b}{cz+d}$ is conformal at all points except $z_0 = -\frac{d}{c} = f^{-1}(\infty)$.

Example Let $f : \Delta \to \Delta$ be any bijective analytic function, $\Delta = B(0,1)$. Claim: $f(z) = \lambda \dfrac{a-z}{1-\bar{a}z}$, $|\lambda| = 1$, $|a| < 1$.

Proof $a = f(0)$. $|a| < 1$

$$\phi(z) = \frac{a-z}{1-\bar{a}z}, \quad \phi : \Delta \to \Delta \text{ bijective.}$$

$$g(z) = \phi(f(z)), \quad g : \Delta \to \Delta \text{ bijective,}$$

$$g(0) = \phi(f(0)) = \phi(a) = 0$$

Schwarz's lemma $\Rightarrow$ $|g(z)| \le |z| \quad \forall z \in \Delta$

$g^{-1} : \Delta \to \Delta$ analytic (!) and bijective, $g^{-1}(0) = 0$

Schwartz's lemma $\Rightarrow$ $|g^{-1}(z)| \le |z|$

$\therefore \quad |z| = |g^{-1}(g(z))| \le |g(z)| \le |z|$

Schwartz lemma $\Rightarrow$ $g(z) = \lambda z$, $|\lambda| = 1$.

$$\frac{a - f(z)}{1 - \bar{a} f(z)} = \lambda z \quad \Rightarrow \quad f(z) = \lambda \frac{\bar{\lambda} a - z}{1 - \lambda \bar{a} z}$$

$\square$

Level Curves

$f : D \longrightarrow \mathbb{C}$ non-constant analytic.

Level curves: $\{ z \in D \mid \text{Re} f(z) = \text{const} \}$

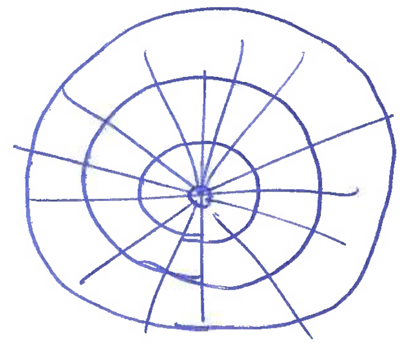
$\{ z \in D \mid \text{Im}(f(z)) = \text{const} \}$.

Example

$f(z) = \log(z) = \ln|z| + i \arg(z)$.

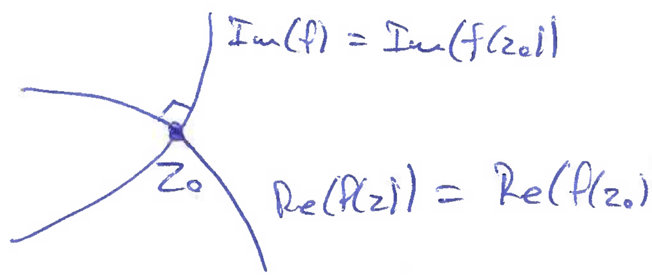
$\text{Re} f(z) = \text{const} \iff |z| = \text{const}$

$\text{Im} f(z) = \text{const} \iff \arg(z) = \text{const}$



Claim

Assume $f'(z_0) \neq 0$. Then the level curves $\{\text{Re} f(z) = \text{Re} f(z_0)\}$ and $\{\text{Im} f(z) = \text{Im} f(z_0)\}$ are perpendicular at z_0.



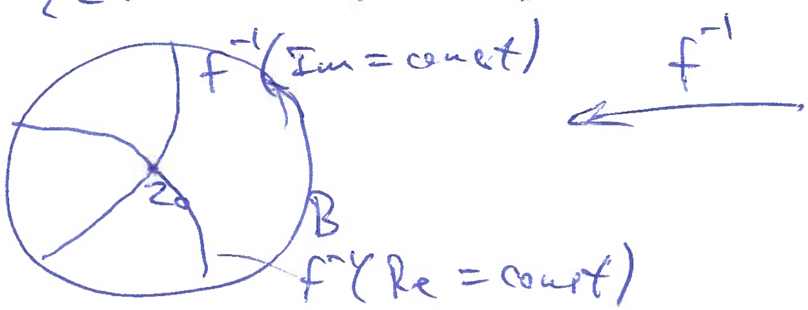
Pf

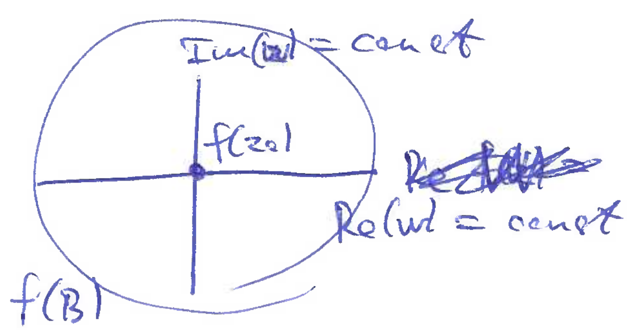
Choose $r > 0$ s.t. $f'(z) \neq 0 \quad \forall z \in B = B(z_0, r)$.

$f : B \longrightarrow f(B)$ one-to-one, analytic.

$f^{-1} : f(B) \longrightarrow B$ 1-1, analytic. So conformal.

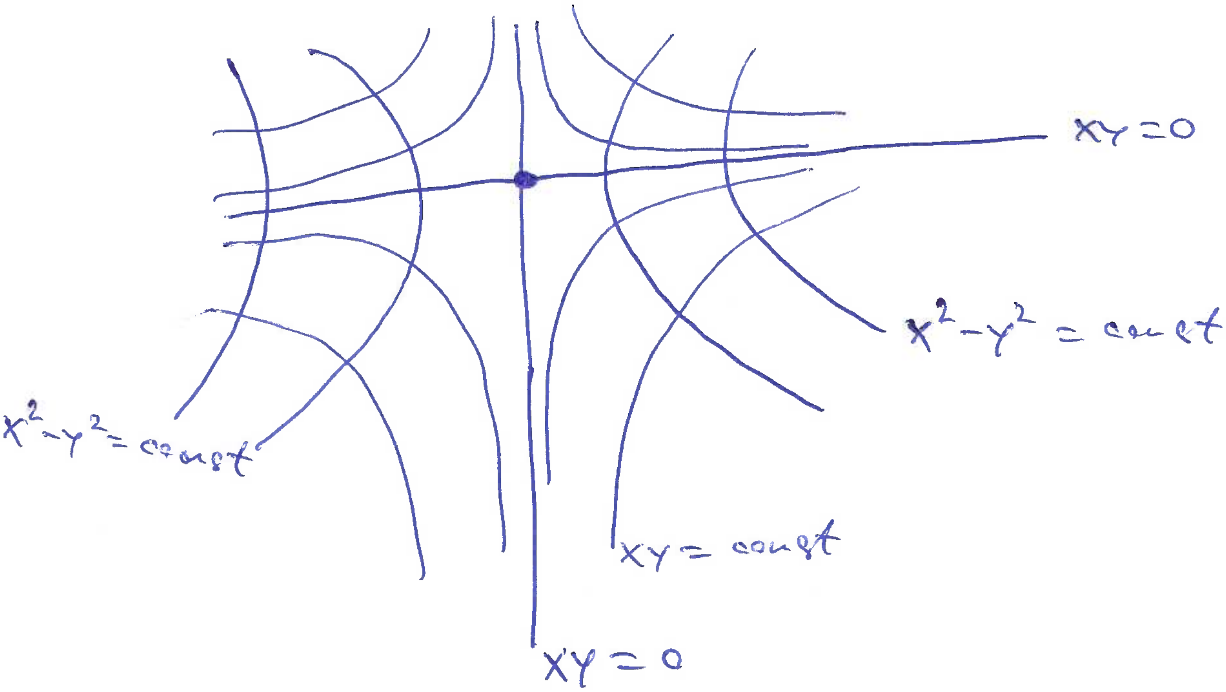
$\{ z : \text{Re} f(z) = c \} =$





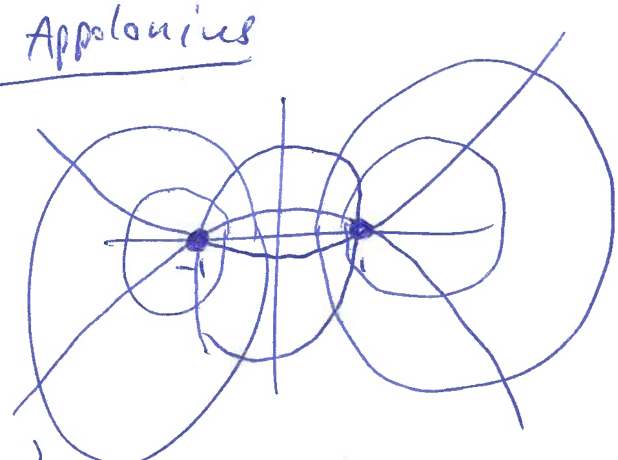
Examples

level curves of $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(xy)$



Example $\quad f(z) = \log\left(\frac{z-1}{z+1}\right)$

level curves are circles of Appolonius



$$\{|z-1| = \rho|z+1|\} \xmapsto{\frac{z-1}{z+1}} \{|w| = \text{const}\} \xmapsto{\log} \{\mathrm{Re}(z) = \text{const}\}$$

$$\{\text{circle through } \pm 1\} \xmapsto{\frac{z-1}{z+1}} \{\text{line through } 0, \infty\} \xmapsto{\log} \{\mathrm{Im}(z) = \text{const}\}$$

Note line = circle through ∞ in $\mathbb{CP}^1$!

Example $f(z) = \sqrt{z} = \exp\left(\frac{1}{2}\log(z)\right)$

$$z = r e^{i\theta}$$

$$f(re^{i\theta}) = \sqrt{r}\cos\left(\frac{\theta}{2}\right) + i\sqrt{r}\sin\left(\frac{\theta}{2}\right)$$

$$\sqrt{r}\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{r}{2}(1+\cos\theta)} = \text{constant}$$

$$\Longleftrightarrow \quad r(1+\cos\theta) = \text{const.}$$

$$\operatorname{Im} f(z) = \text{const} \quad \Longleftrightarrow \quad r(1-\cos\theta) = \text{const.}$$